Message-passing on hypergraphs: detectability, phase transitions and higher-order information

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PAPER • OPEN ACCESS

Message-passing on hypergraphs: detectability, phase transitions and higher-order information

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JSTAT paper

arXiv 2312.00708

GitHub

Keywords

- Hypergraphs
- Community detection
- Message passing

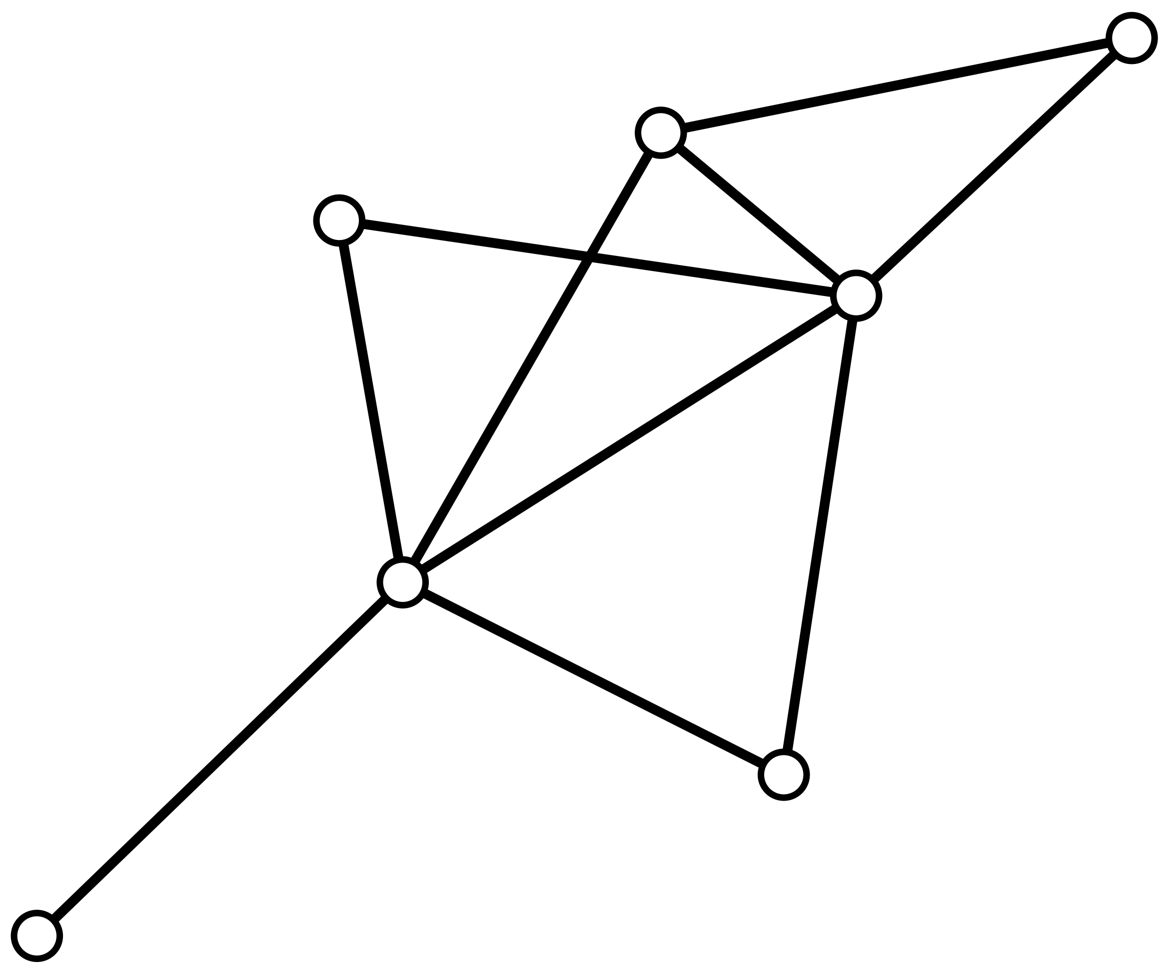
Warning

The paper is rather technical. What matters is understanding / getting an intuition of:

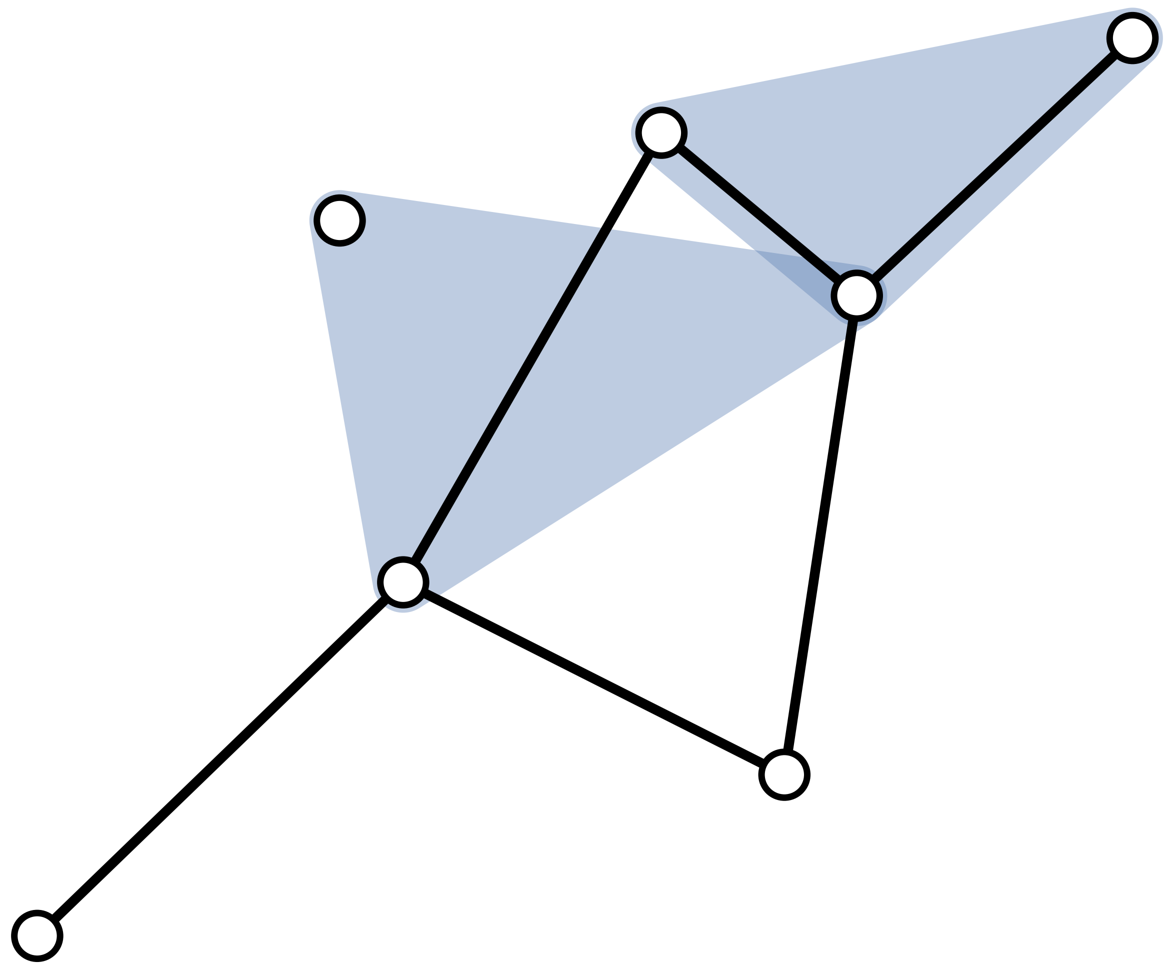
- The problem
- The challenges
- The result

Slides with difficult content are marked with 🔴

Graph

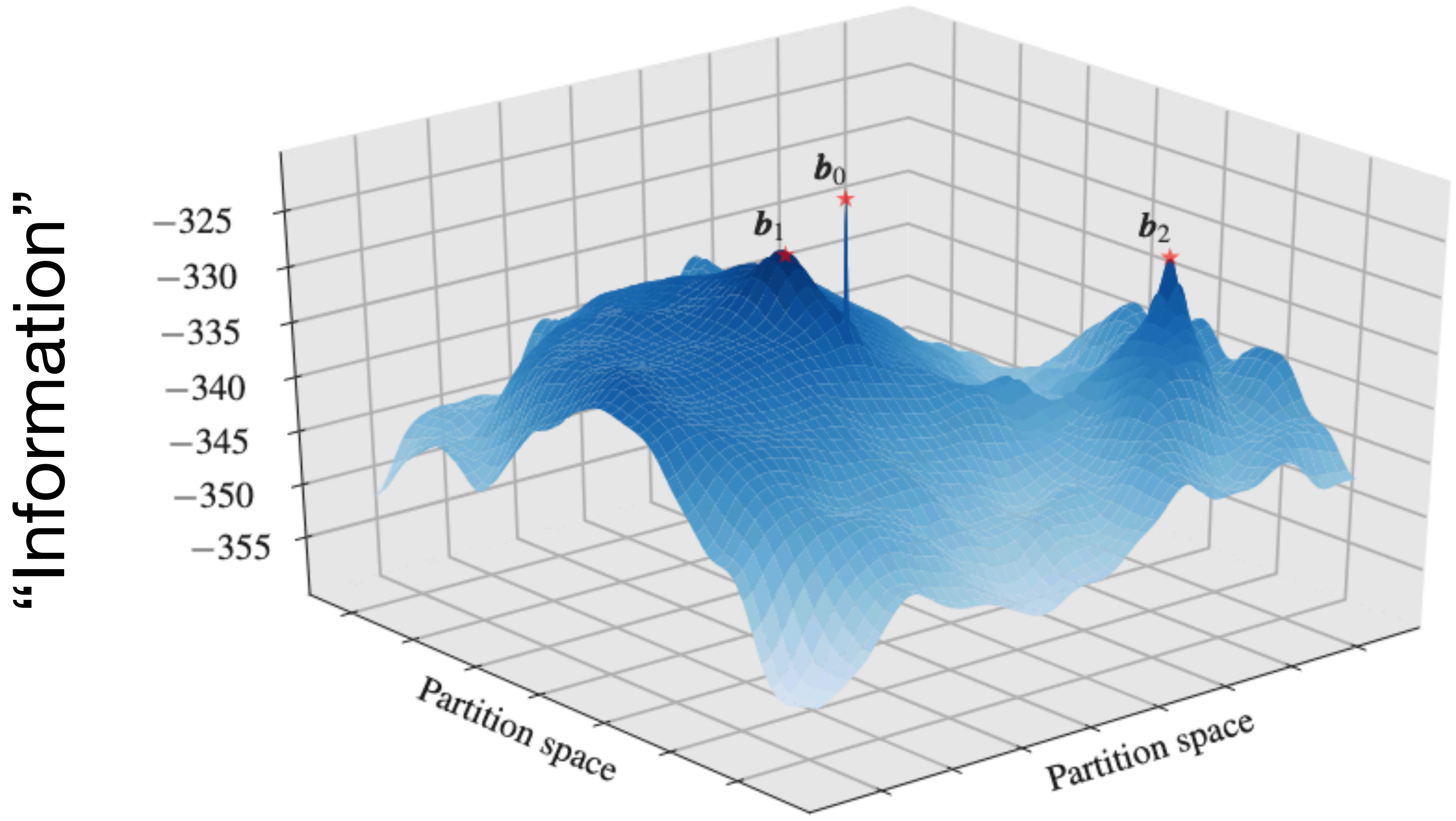


Hypergraph



- Hypergraphs and graphs **ARE** different objects
    - A hypergraph induces a unique clique decomposition but the converse is not true
    - Daily life intuition: group chat vs. direct message
    - [Battiston et al. Physics Reports 874 (2020)](#)

Community detection



[Peixoto, Advances in Network Clustering and Blockmodeling (2019)]

Community detection is hard

- Ground truth communities are difficult or impossible to define
  [Peel et al. Sci. Adv. 3, e1602548 (2017)]
- How does one measure how much “Information” the communities give?
- The problem has exponential computational complexity, solutions have to be efficient

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Methods

- Descriptive / inferential community detection
  Schaub et al. Applied Network Science (2017)

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Methods

- Descriptive / inferential community detection
  [Schaub et al. Applied Network Science (2017)]

Message passing (coming next) falls in this category

Message passing

Message passing is a method to perform inference on graphical models

Mézard and Montanari, Information, Physics, and Computation (2009)

Message passing

Message passing is a method to perform inference on graphical models

- Community detection
- Factor graph: a graph to represent the factorization of a function, the probability of belonging to a community given the graph / hypergraph

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It is an efficient method on graphs!

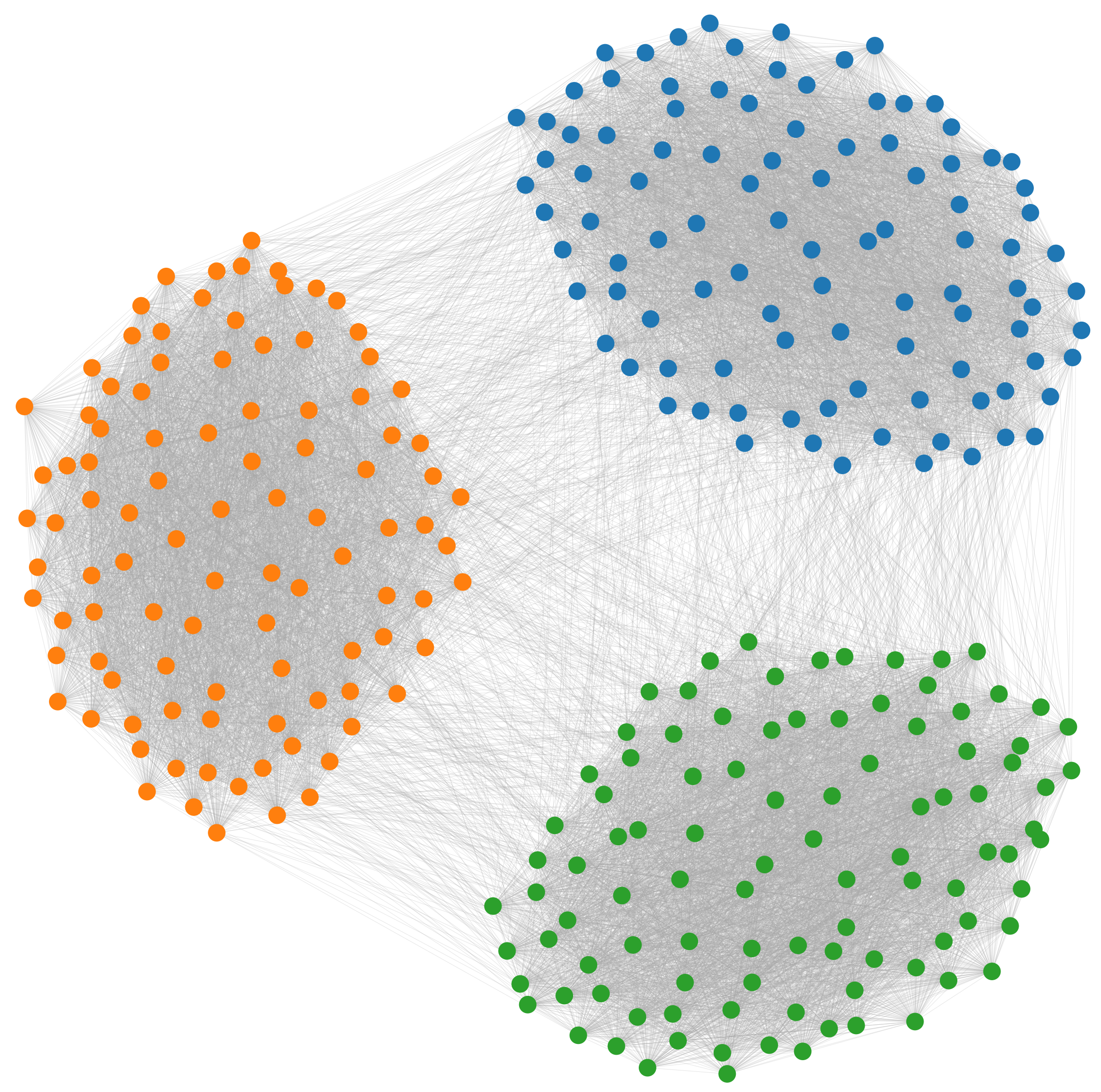
Paying the price of an approximation, we reduce the complexity **from exponential** to **polynomial**

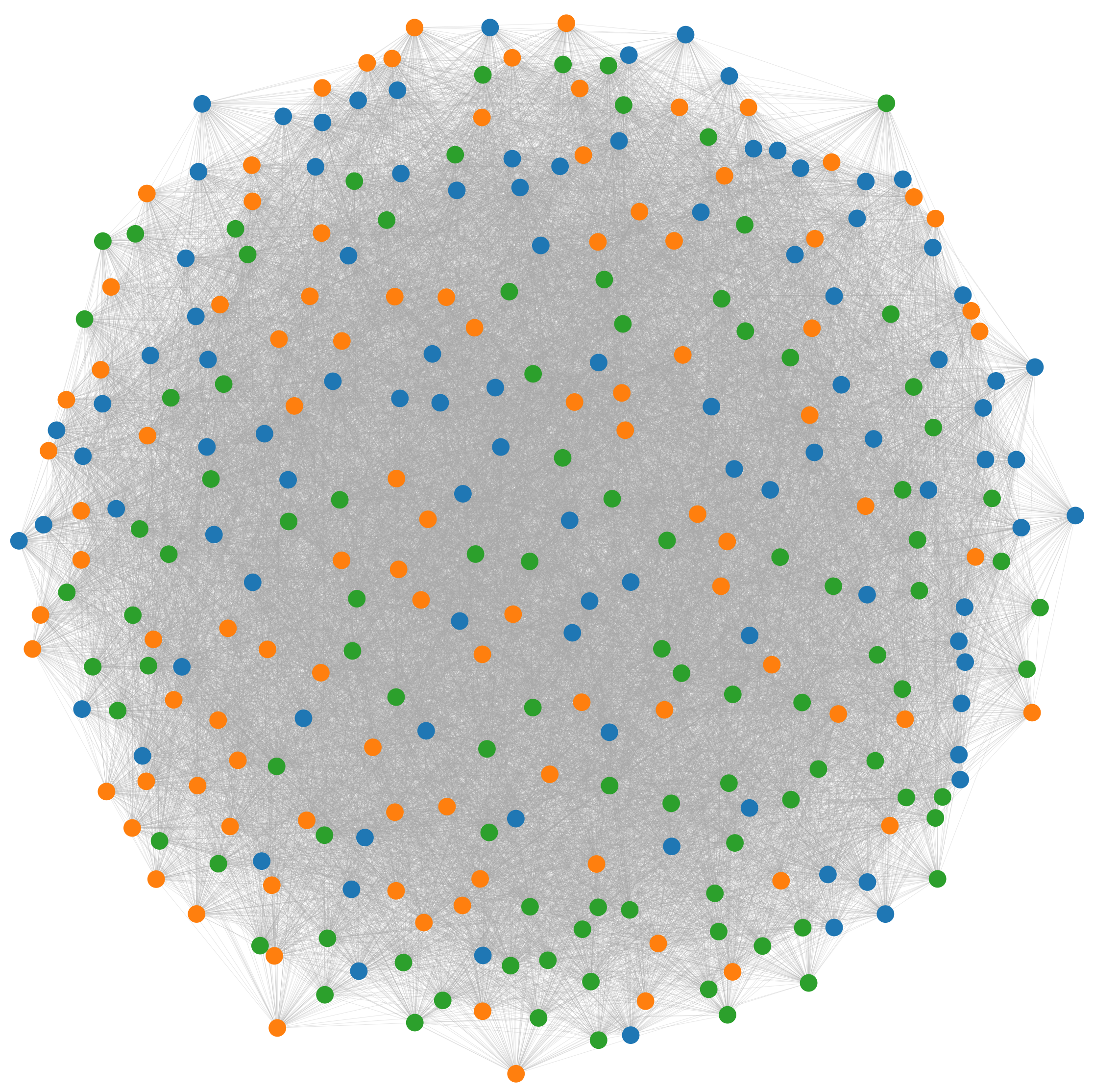
Let's look at how inferential community detection works in practice

A famous result

Task: Stochastic Block Model

- Sample the ground truth communities
- Build a graph where $i \sim j$ in class a, b is a random variable $A_{ij} \sim \text{Bernoulli}(p_{ab})$

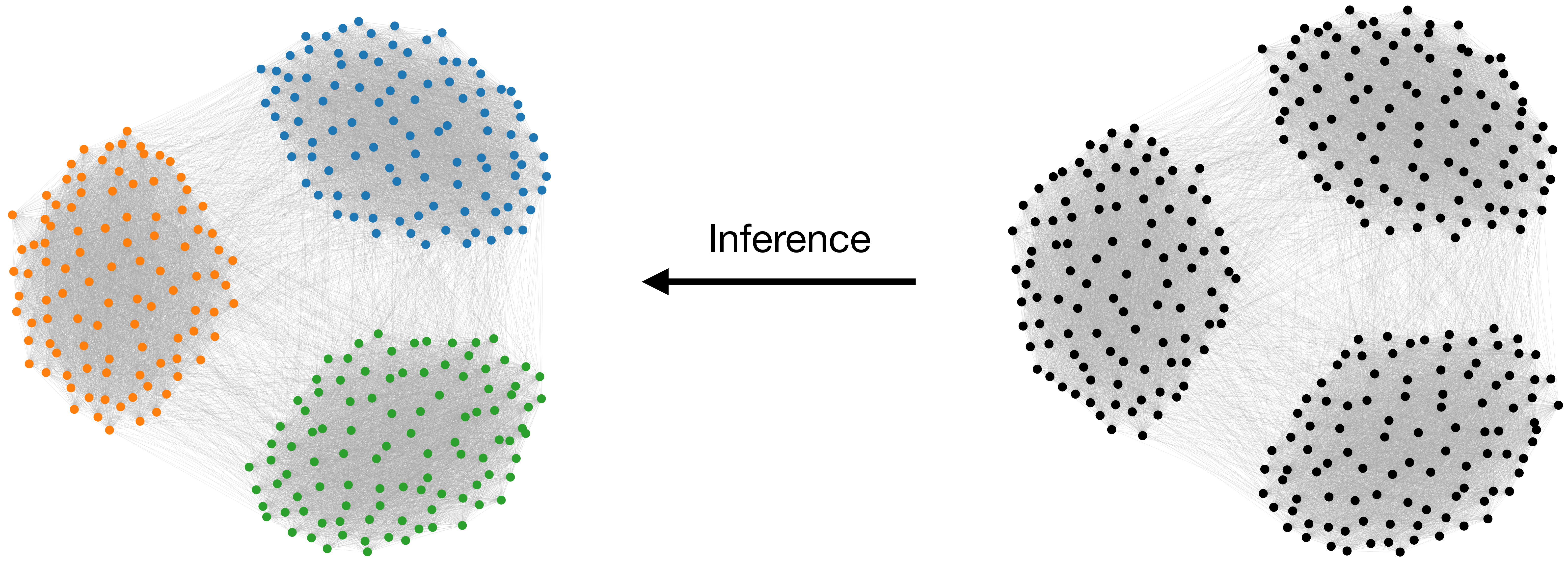




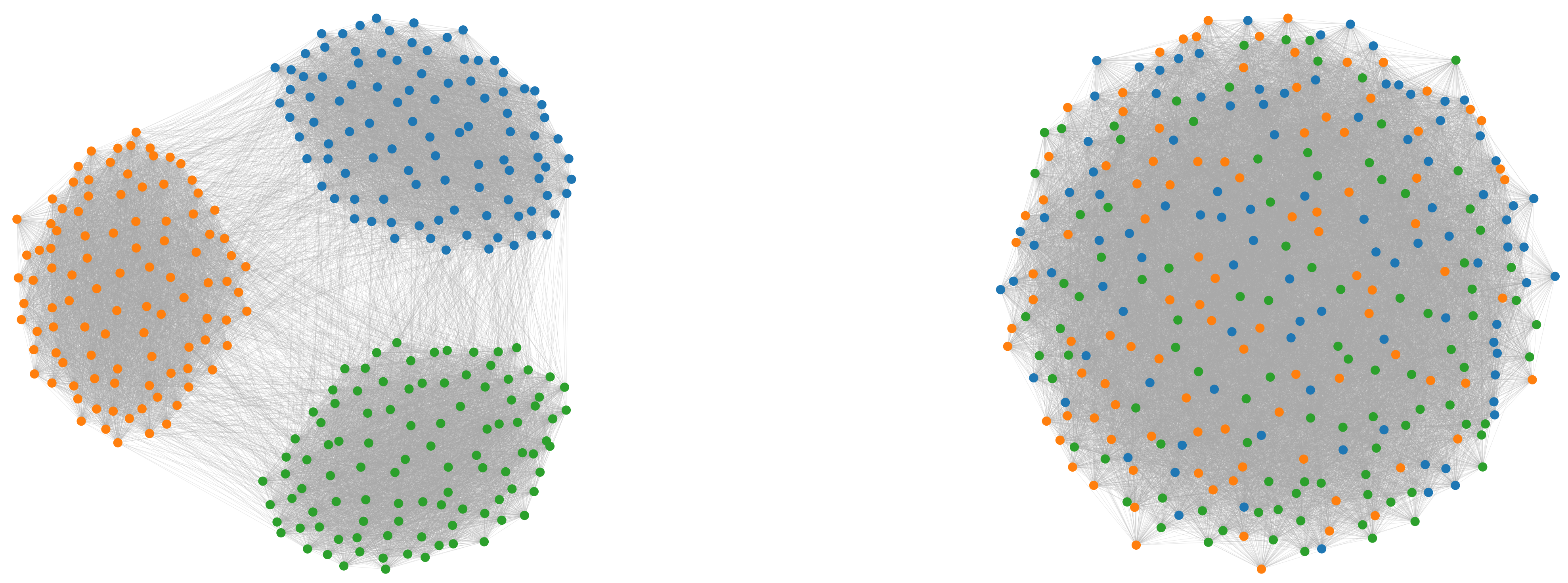
A famous result

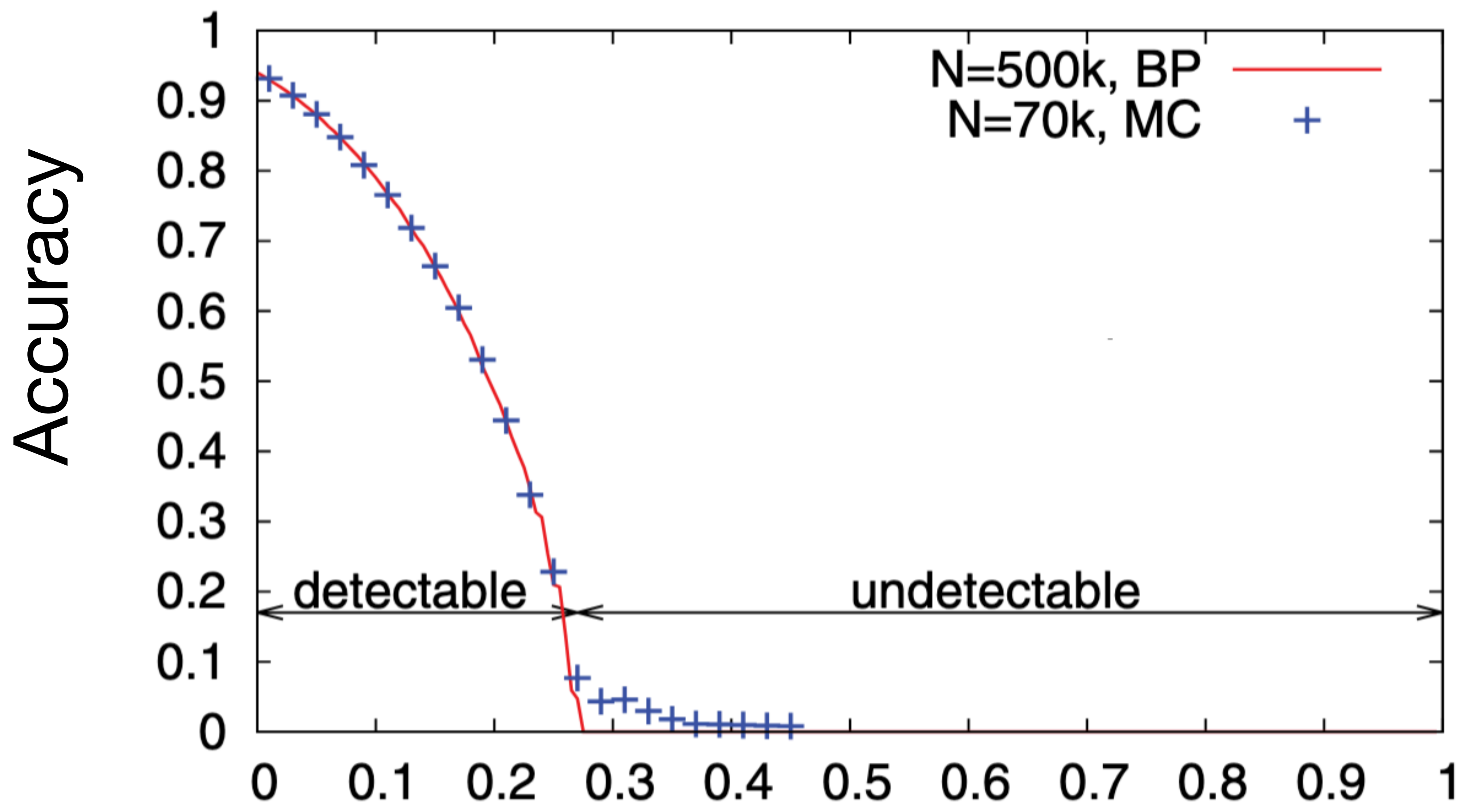
Task: Stochastic Block Model

- Sample the ground truth communities
- Build a graph where $i \sim j$ in class a, b is a random variable $A_{ij} \sim \text{Bernoulli}\,(p_{ab})$
- Detect communities (with MP, you get the probability of a node belonging to community)



A famous result





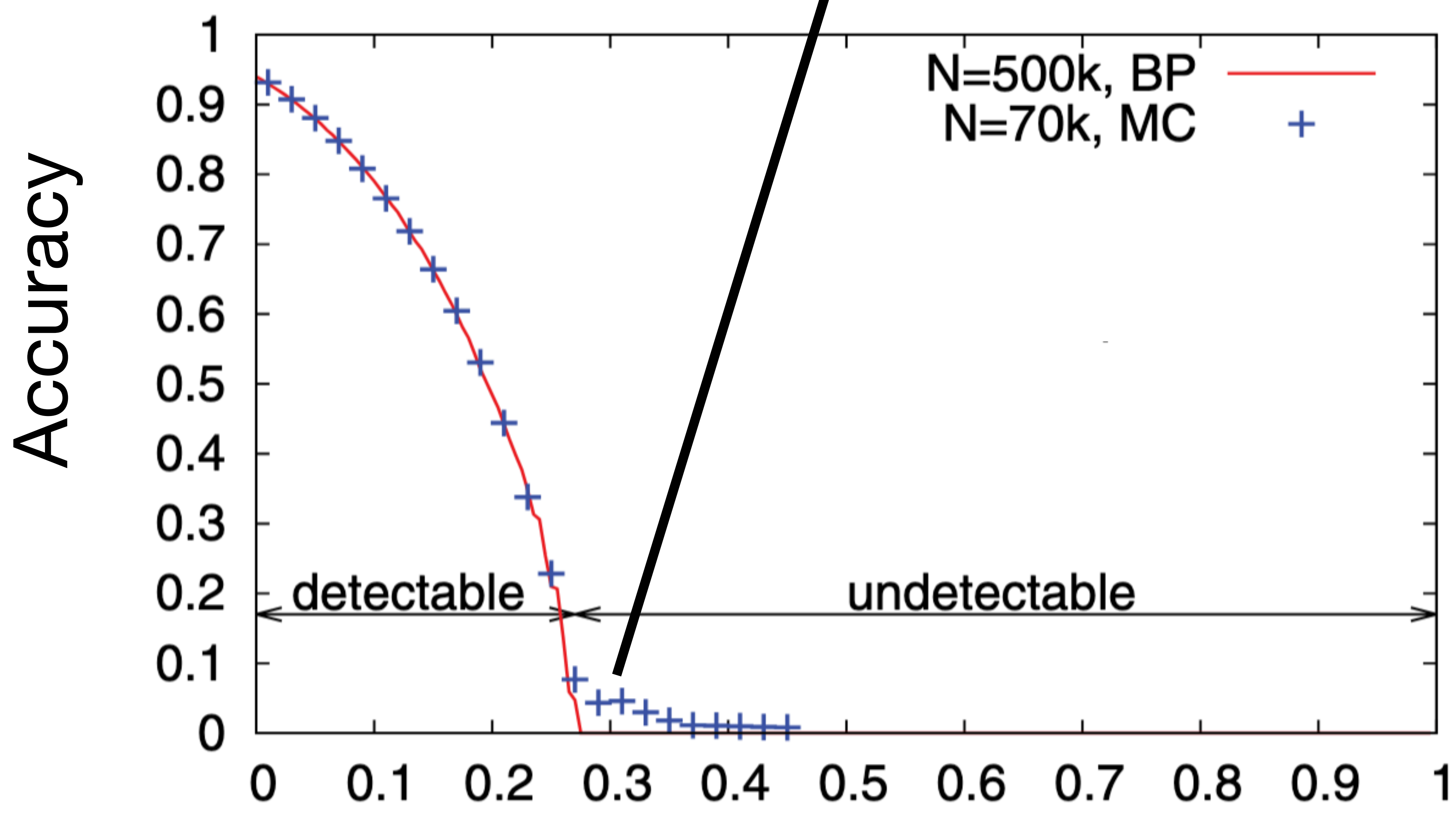
Assortative ⟵ in degree / out degree ⟶ Mixed up

[Decelle et al. PRL (2011)]

A famous result

$$N \to \infty$$

$$|\text{in degree - out degree}| = \#\text{communities}\sqrt{\text{avg. degree}}$$



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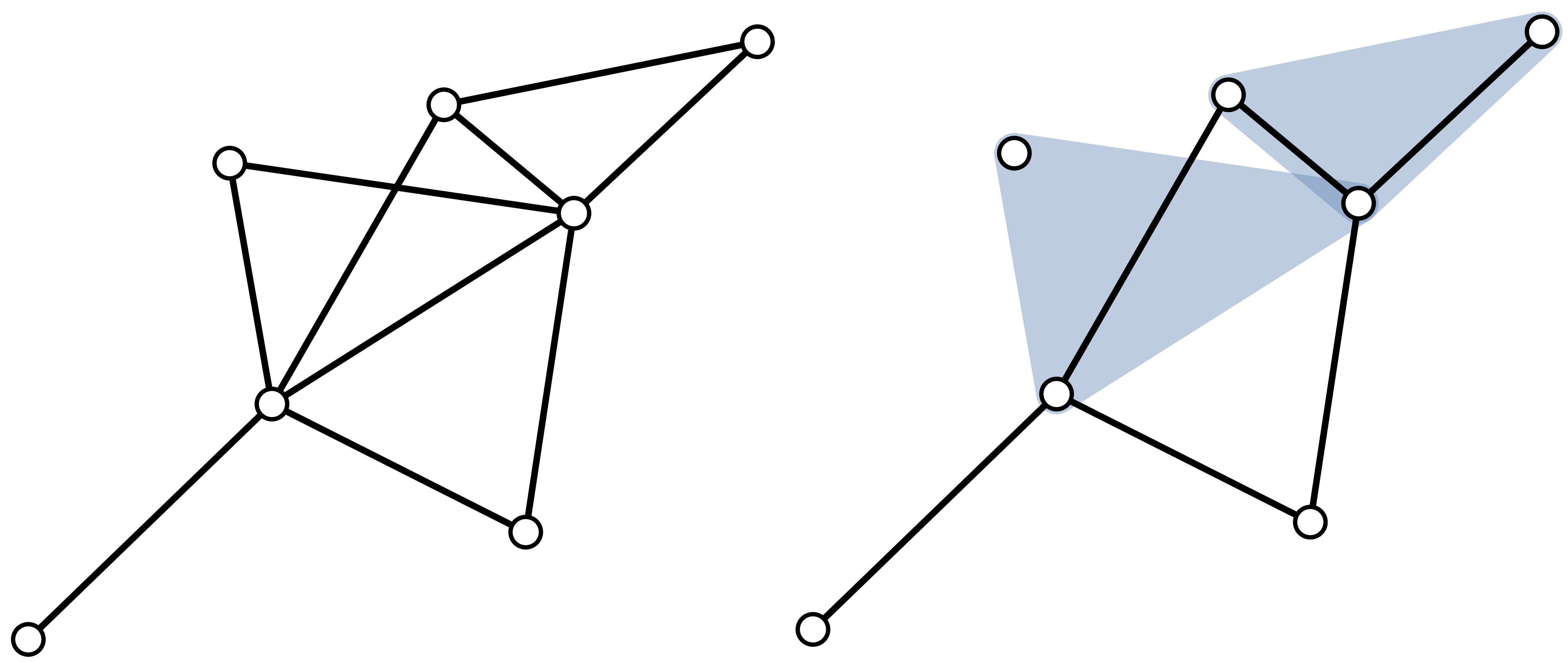
What about hypergraphs?



Does a phase transition appear in hypergraphs?

If yes,

1. What is the contribution of hyperedges?
2. Can we find the critical threshold in closed form?
3. Will the result tell us something important about hypergraphs?

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Task: Stochastic Block Model

- Sample the ground truth communities
- Build a hypergraph where A_e is a random variable $A_e \sim$ Bernoulli $\left(\frac{1}{\kappa_e} \sum_{i<j \in e} p_{c(i)c(j)} \right)$

- Build a graph where $i \sim j$ in class a, b is a random variable $A_{ij} \sim$ Bernoulli (p_{ab})

Task: Stochastic Block Model

- Sample the ground truth communities
- Build a hypergraph where A_e is a random variable $A_e \sim \text{Bernoulli}\left(\frac{1}{\kappa_e}\sum_{i<j\in e} p_{c(i)c(j)}\right)$
- Build a graph where $i \sim j$ in class a, b is a random variable $A_{ij} \sim \text{Bernoulli}\,(p_{ab})$
- Detect communities 🔴

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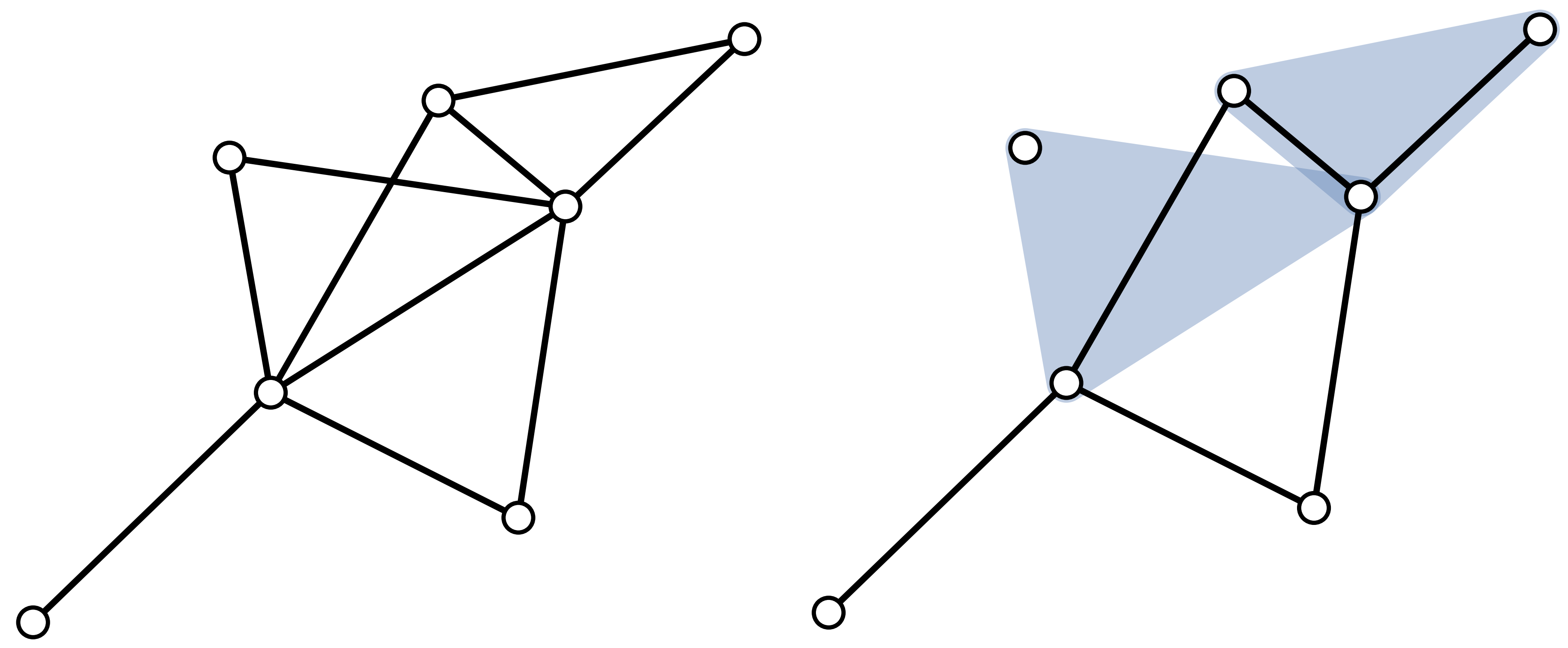
Let's hone the intuition

Message passing on a factor graph gives us the probability of i to be in a community a based on "its neighbors" (a cavity)



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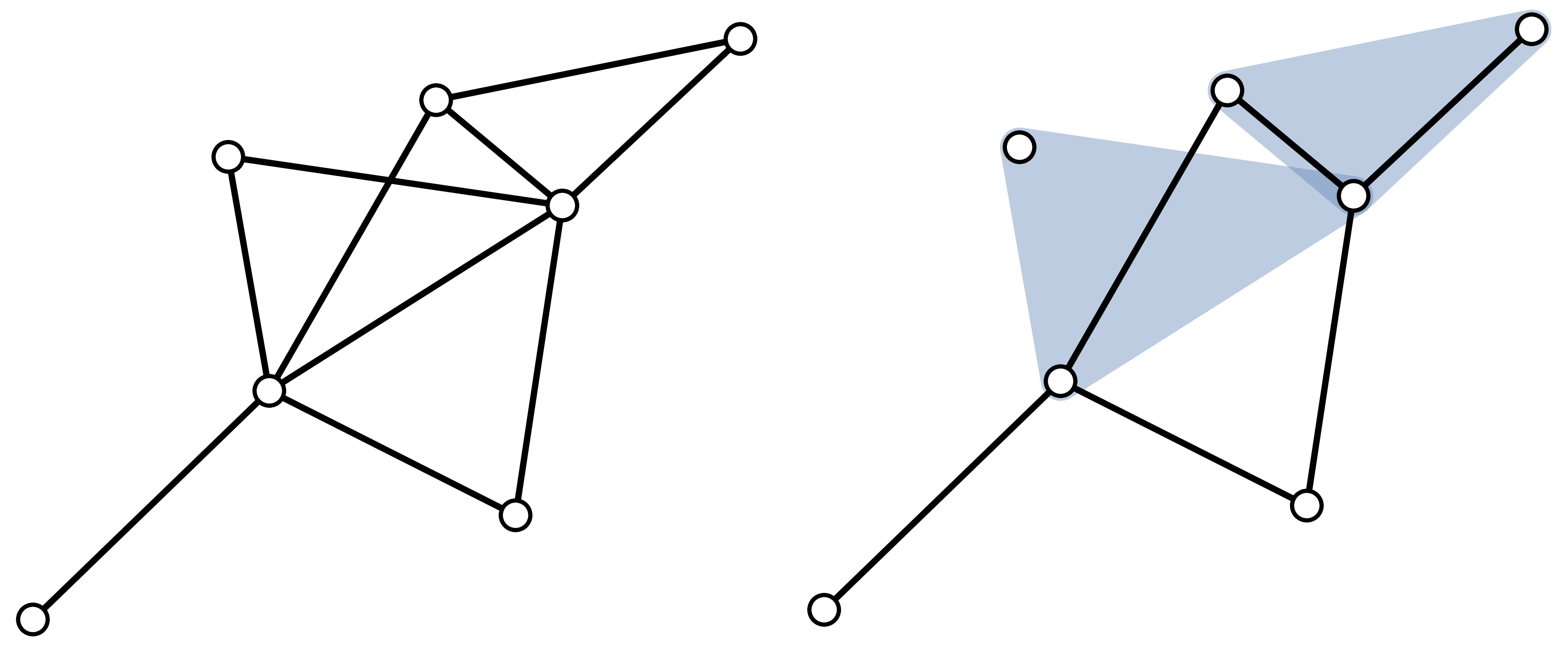
Challenge: additional exponential complexity



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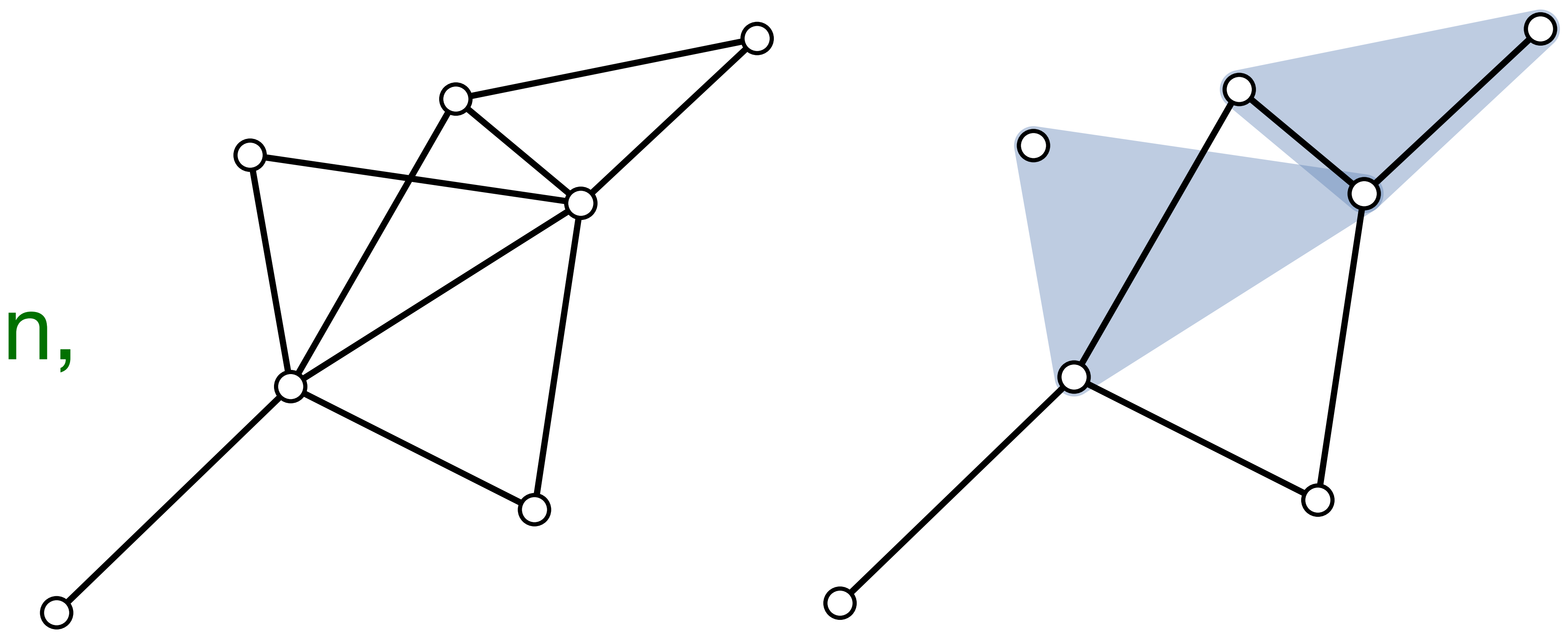
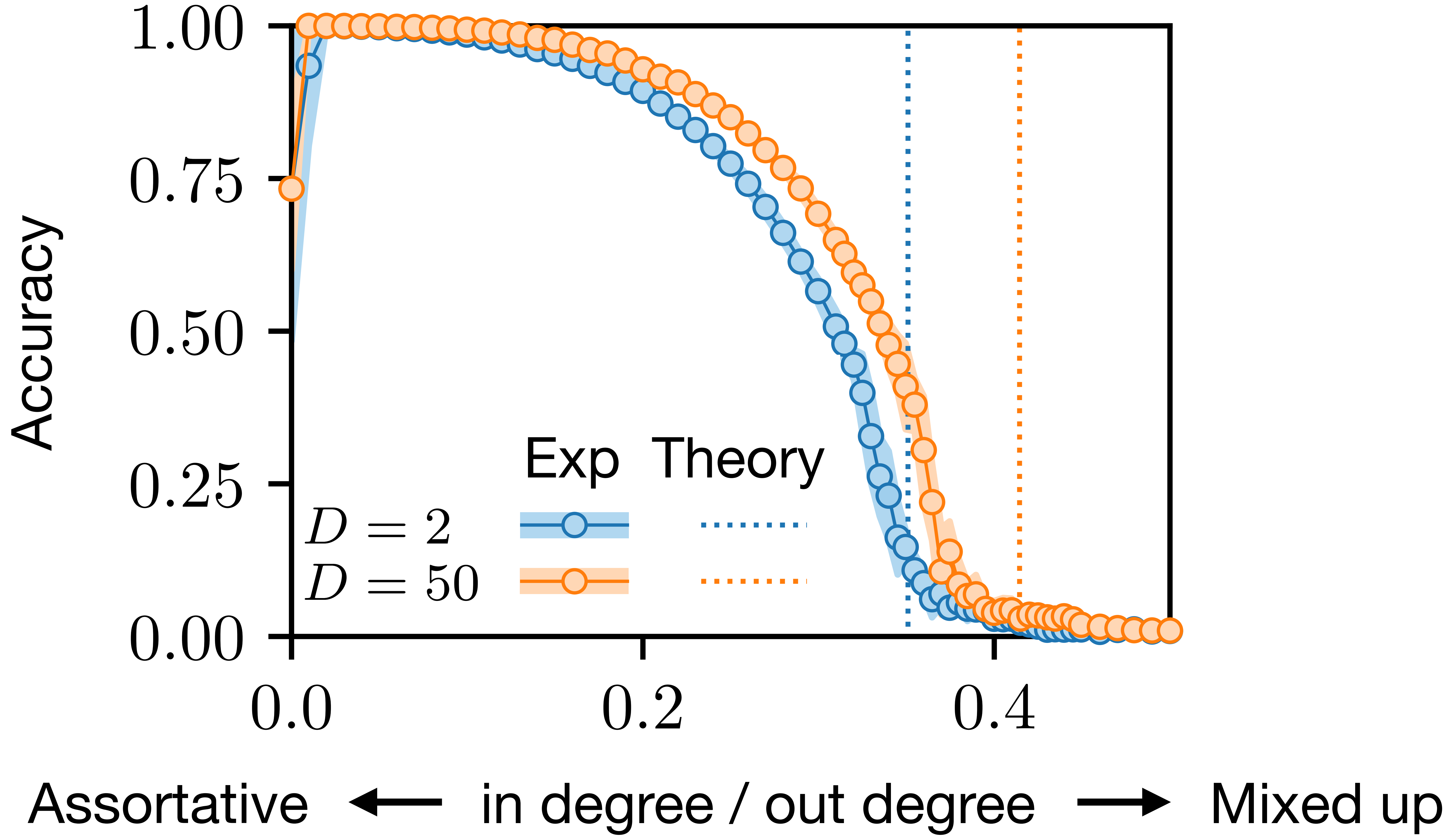
Challenge: additional exponential complexity

Solution: dynamic programming, no approximation, **polynomial complexity**



Phase transition for hypergraphs



Ruggeri, AL, De Bacco JSTAT (2024)

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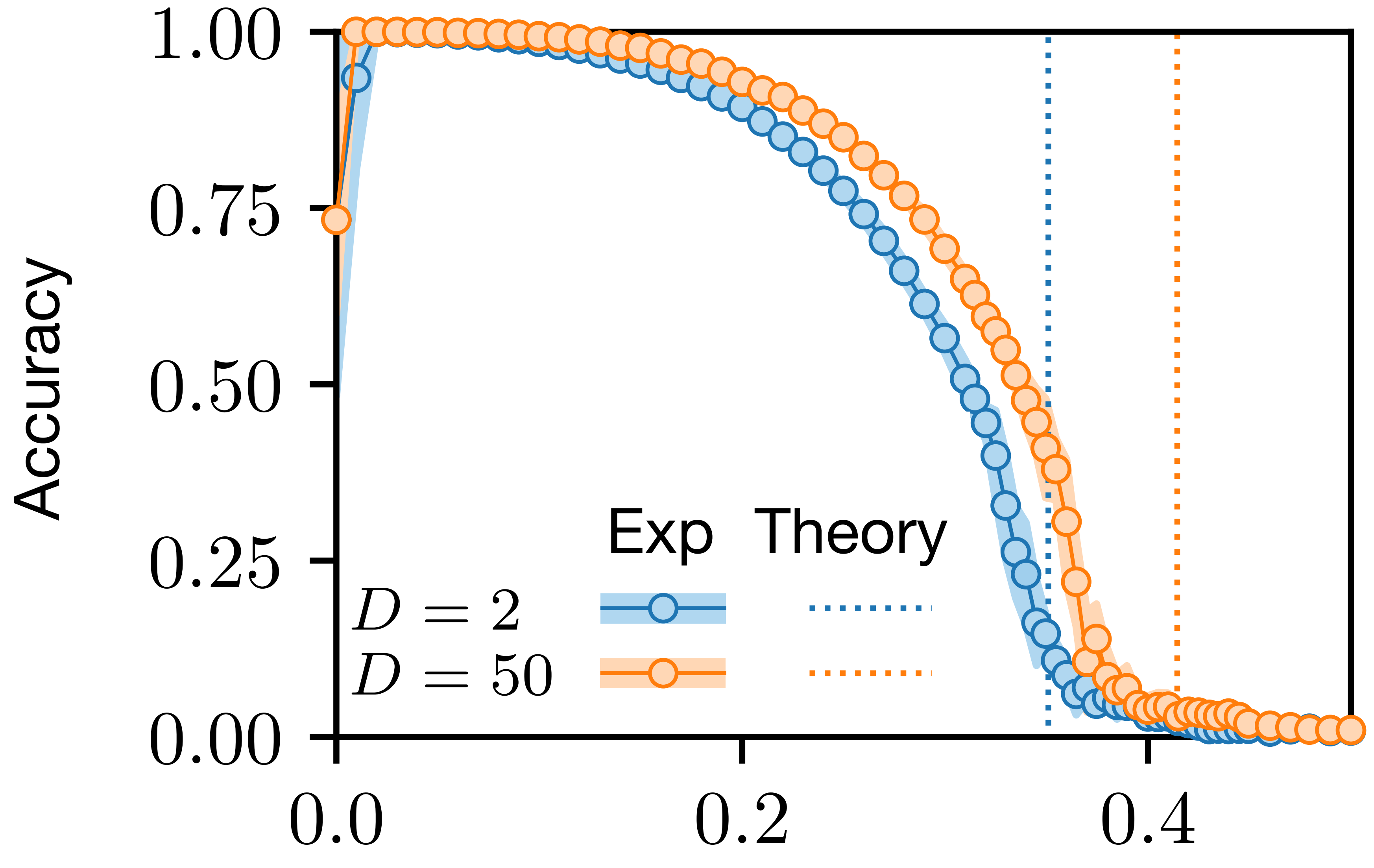

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$N \to \infty$

$|\,\text{in degree - out degree}\,| = \#\text{communities}\sqrt{\text{avg. degree}}$

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$$g(\text{hyperedge distribution})$$

(the lower the better, the phase transition shifts to the right)

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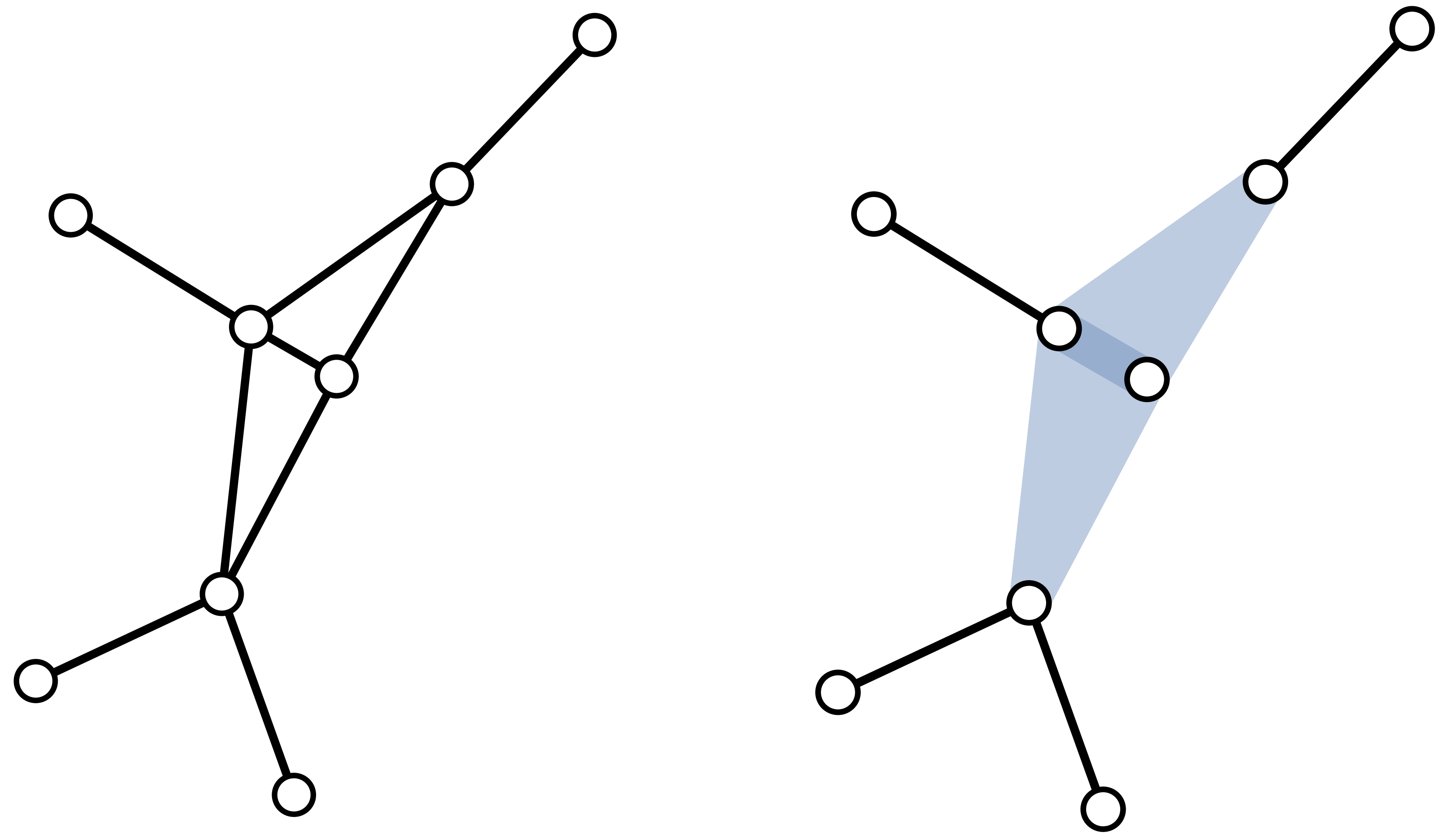

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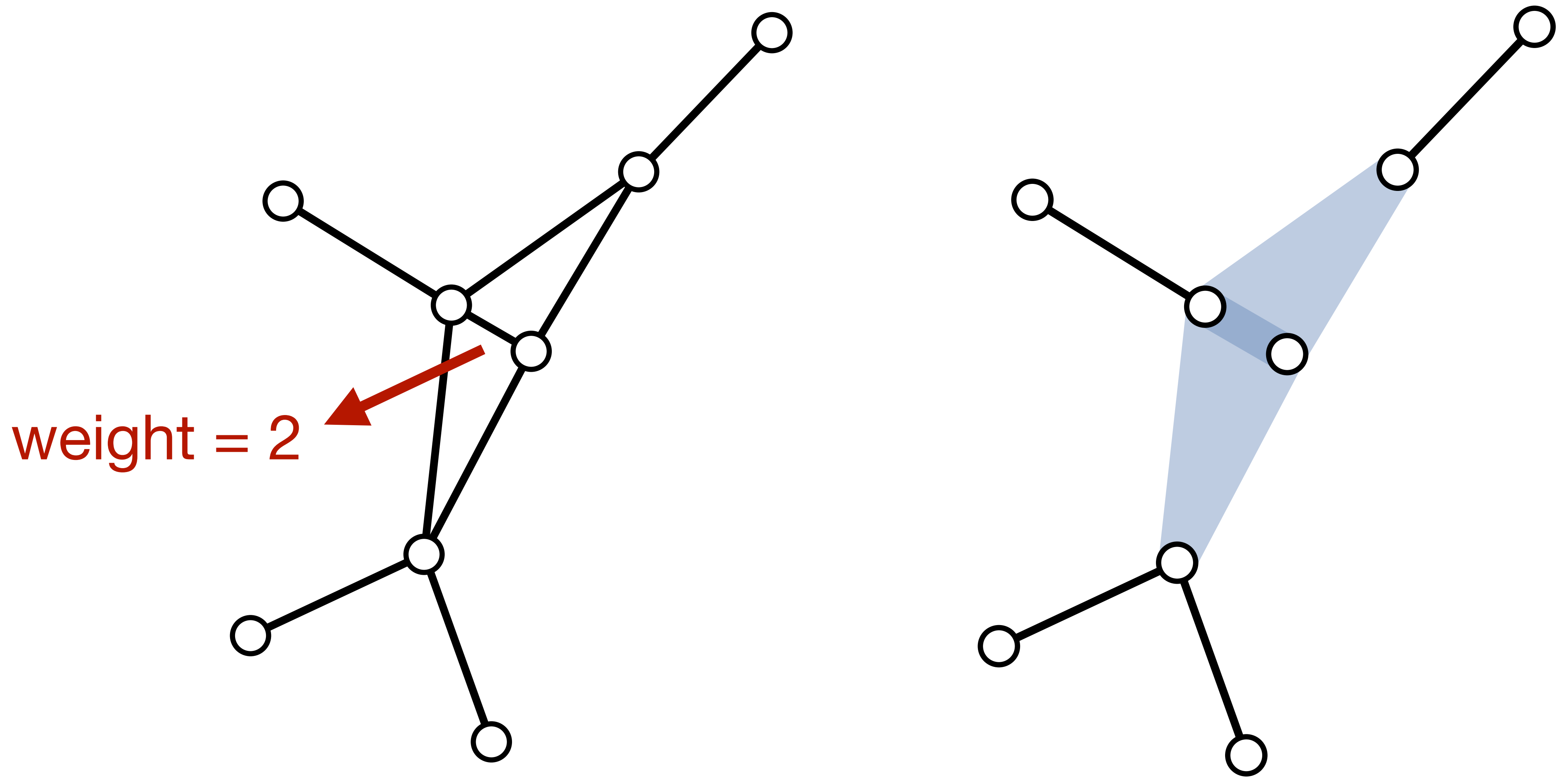

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[Ruggeri, AL, De Bacco JSTAT (2024)]

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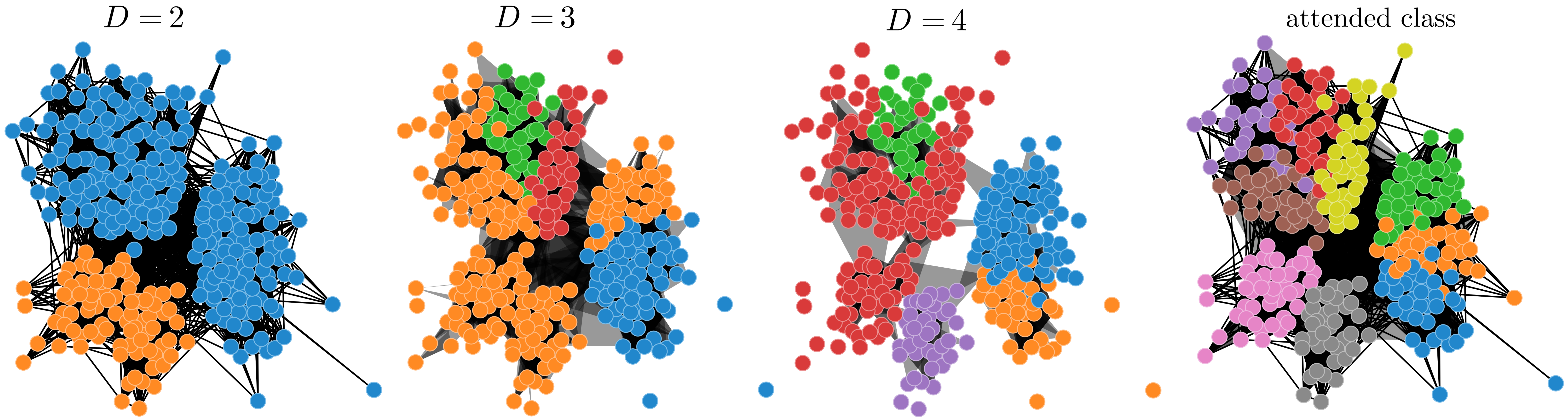
$$\boxed{g = H(\{i,j\}) - \mathrm{KL}(p_H || p_C \otimes p_E)}$$

$$\mathrm{KL}(p_H || p_C \otimes p_E) = I(e, \{i,j\})$$

Ruggeri, AL, De Bacco JSTAT (2024)

Social contact



[Ruggeri, AL, De Bacco JSTAT (2024)](#)

Take home message

Principled and efficient inference on large complex systems is possible…

… and we should do it

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