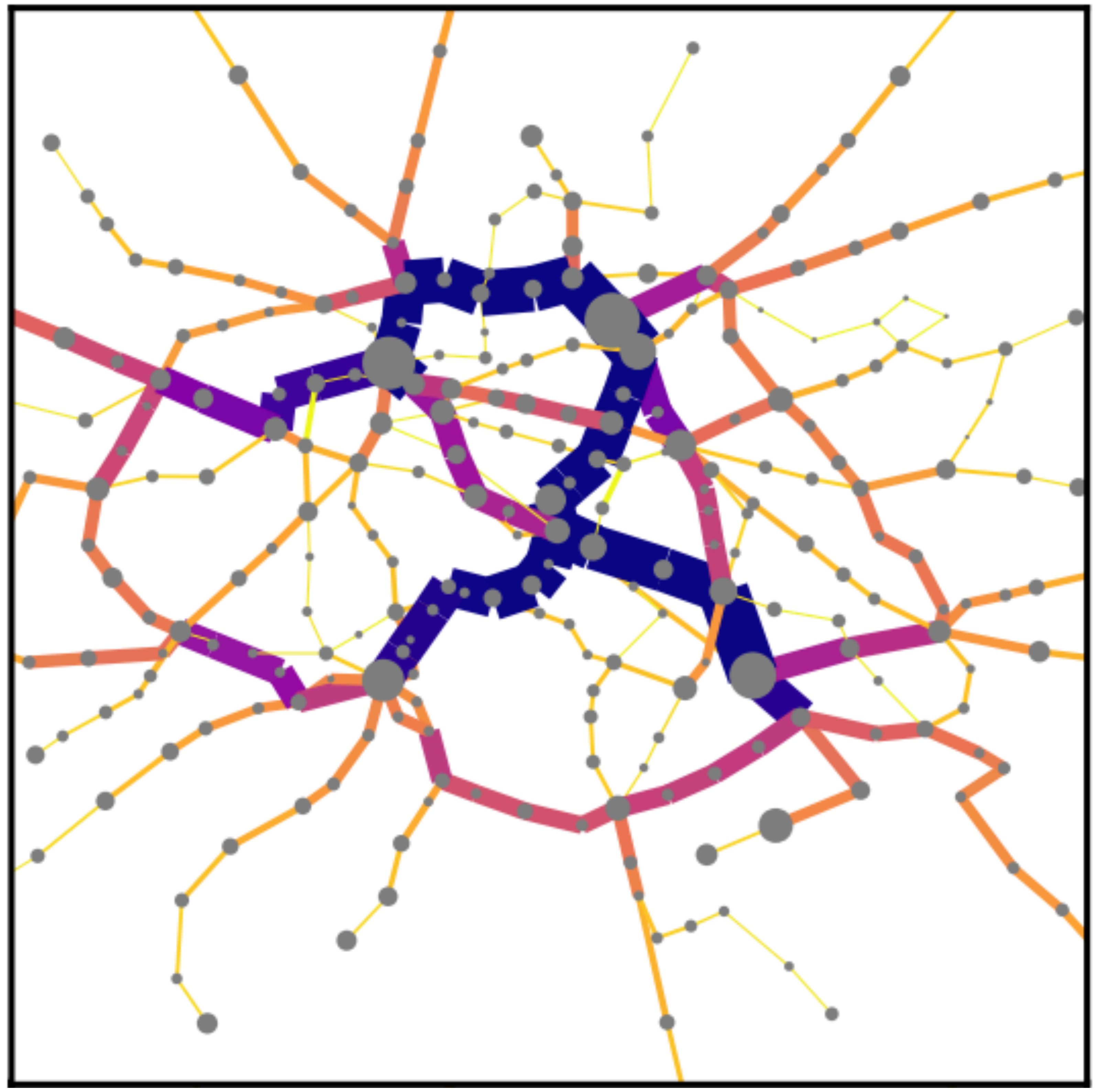
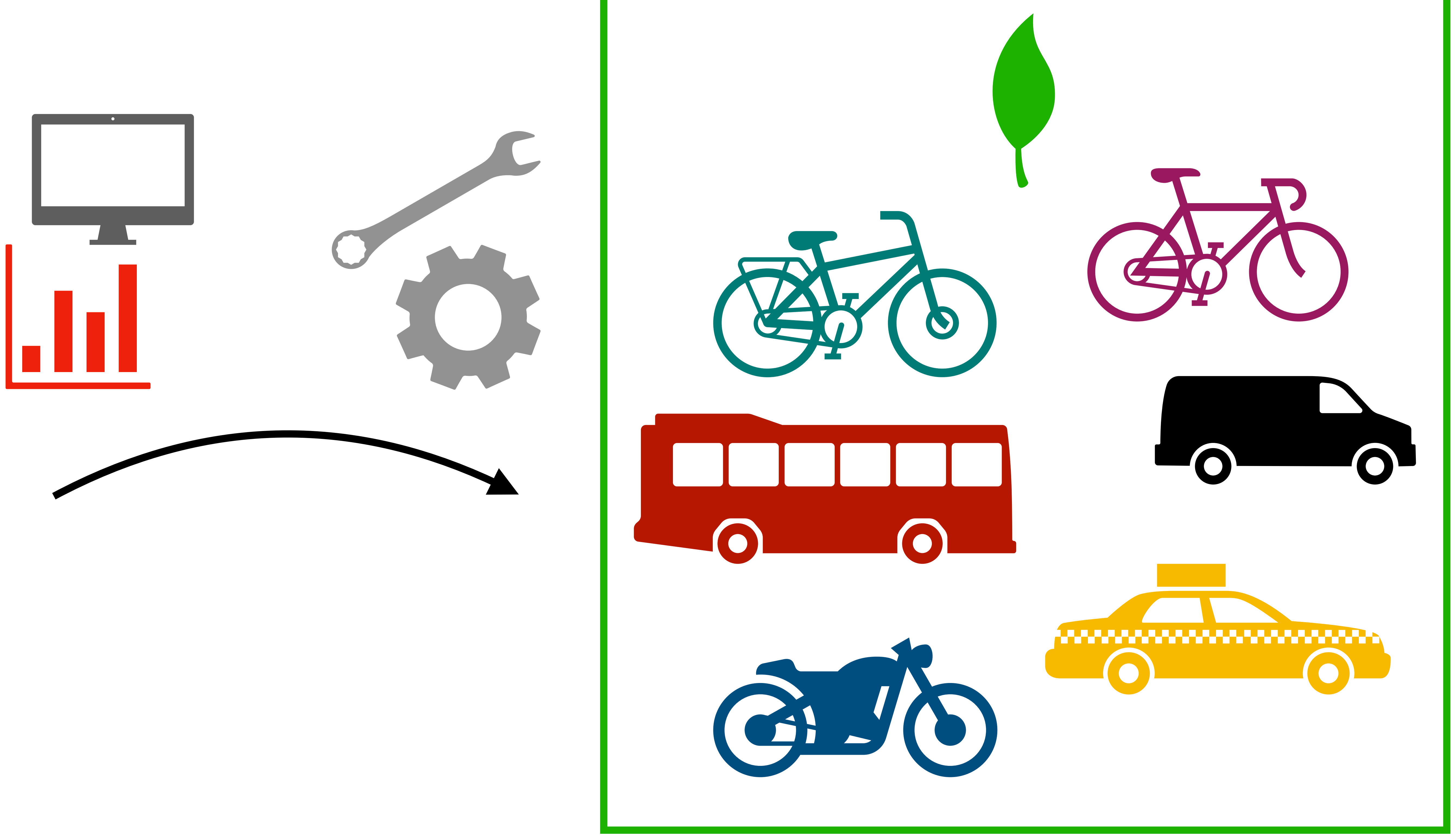
Optimal transport in networks for design and flux optimization

NetPLACE Seminars — 9th March 2023

Alessandro Lonardi
Physics for Inference and Optimization, MPI IS Tübingen





MPI IS & Physics for Inference and Optimization

MAX-PLANCK-GESELLSCHAFT

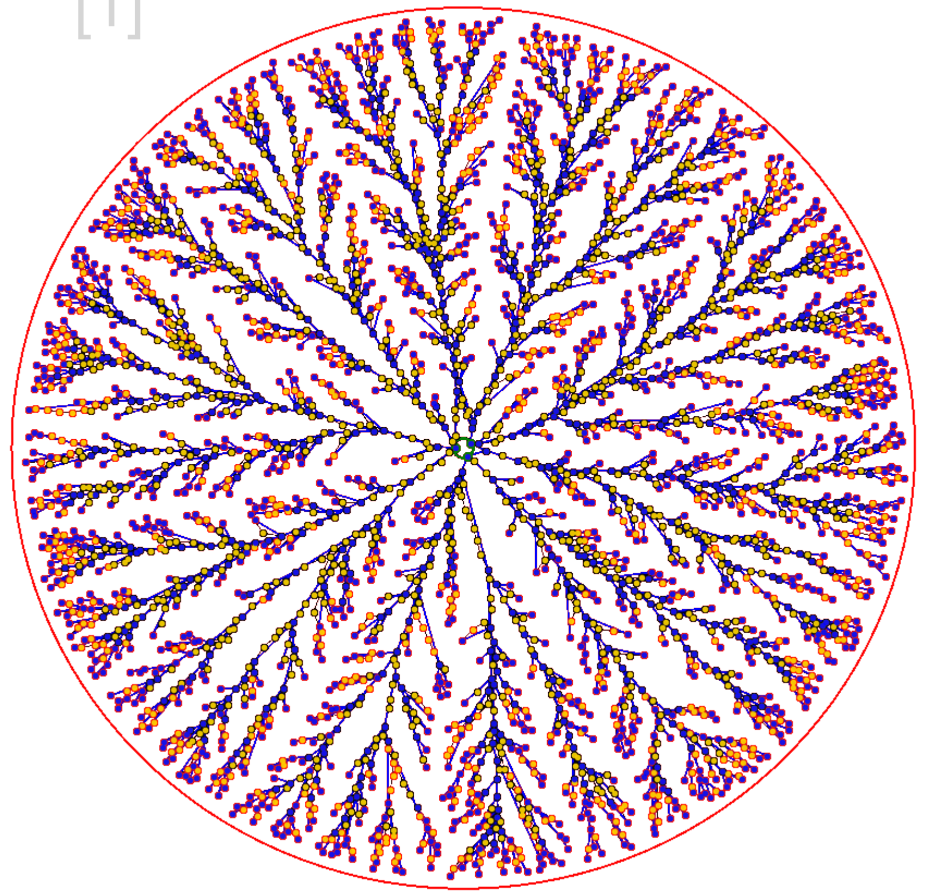
impres-is

Cyber Valley

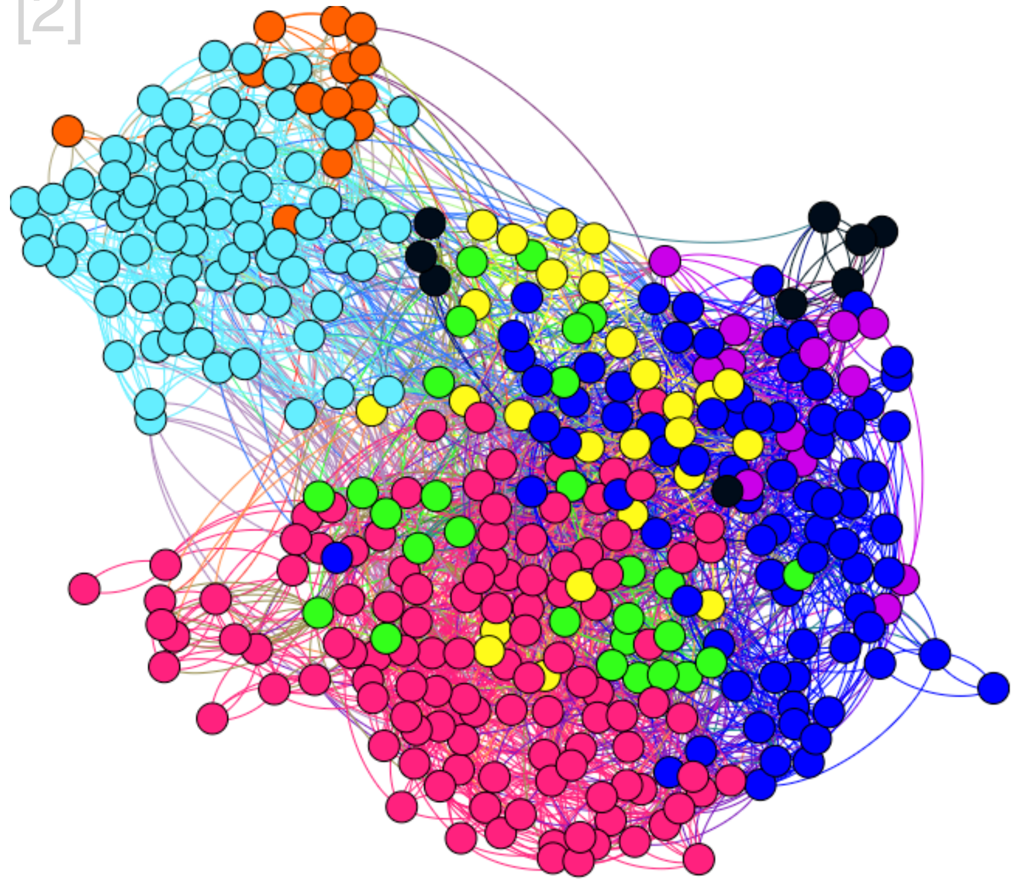
Alessandro Lonardi — MPI IS— NetPLACE — 9th March 2023

Physics for Inference and Optimization

[1]



[2]



🛠️ Statistical Physics, probability, statistics, mathematics, CS, etc.

🎯 Networks: community detection, network inference, network routing, etc.

[1] Baptista et al. Sci. Rep. 2020
[2] De Bacco et al. PRE 2017

Physics for Inference and Optimization

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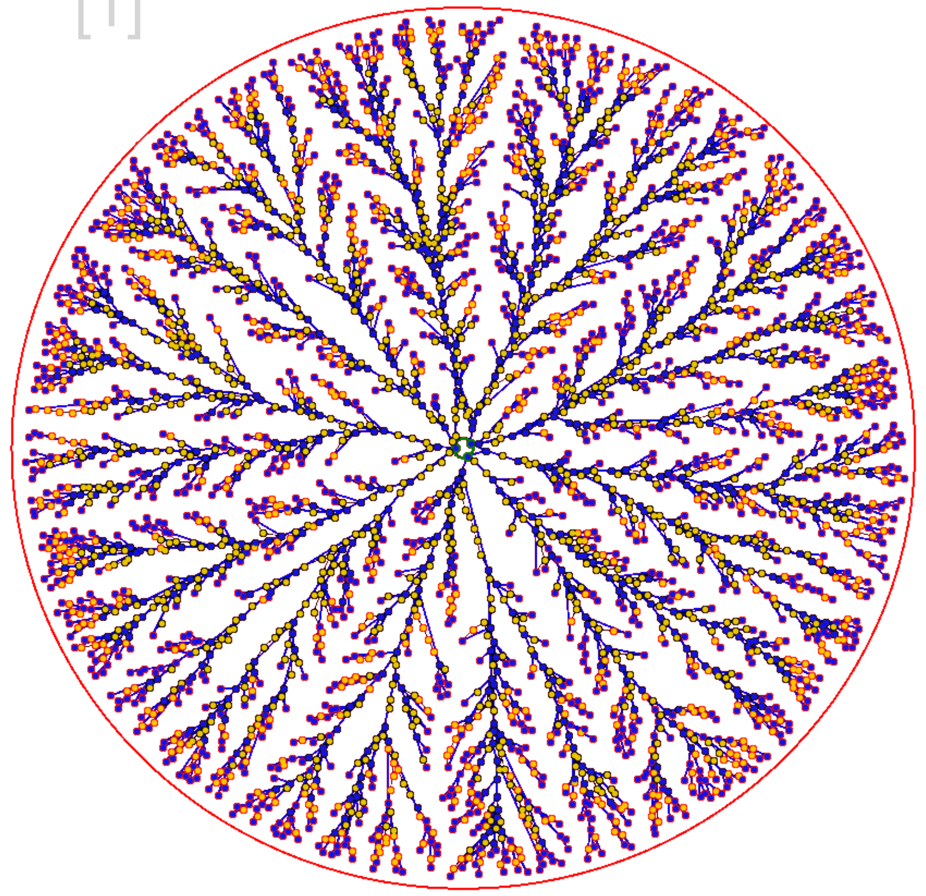

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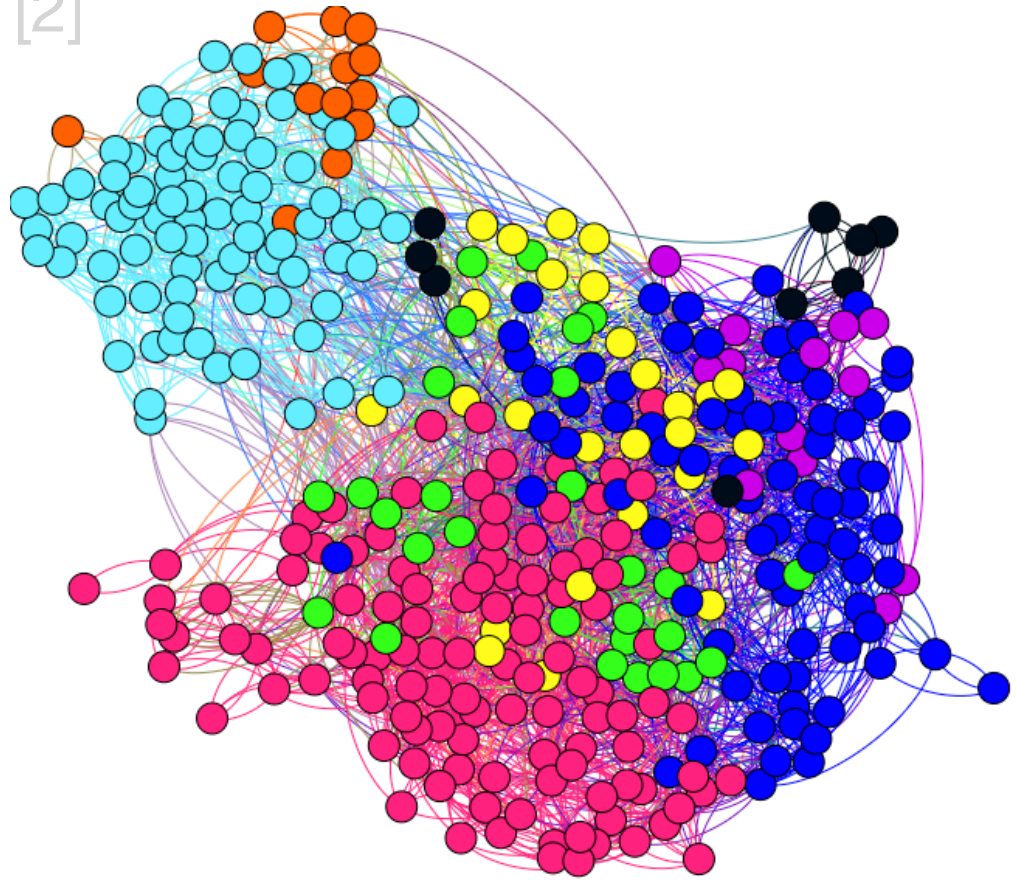

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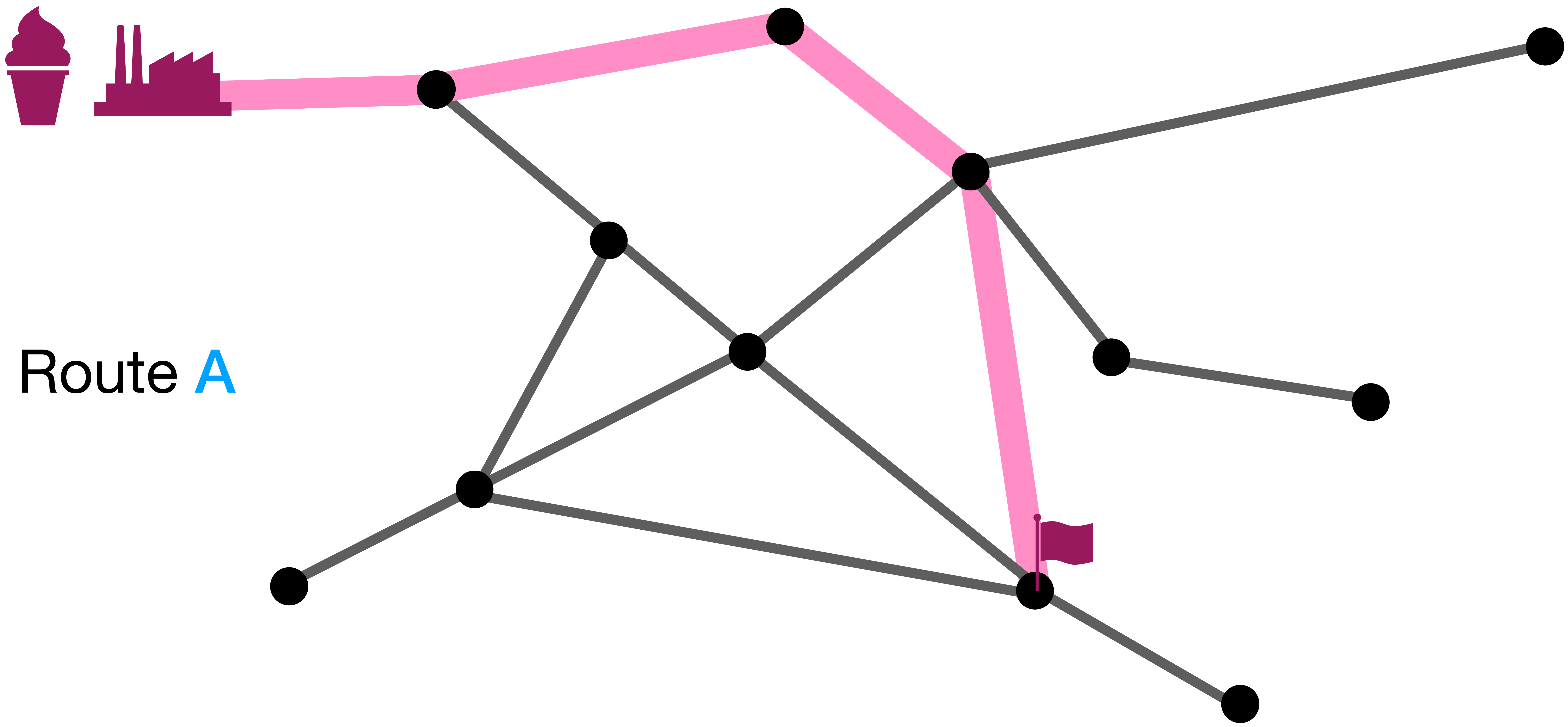
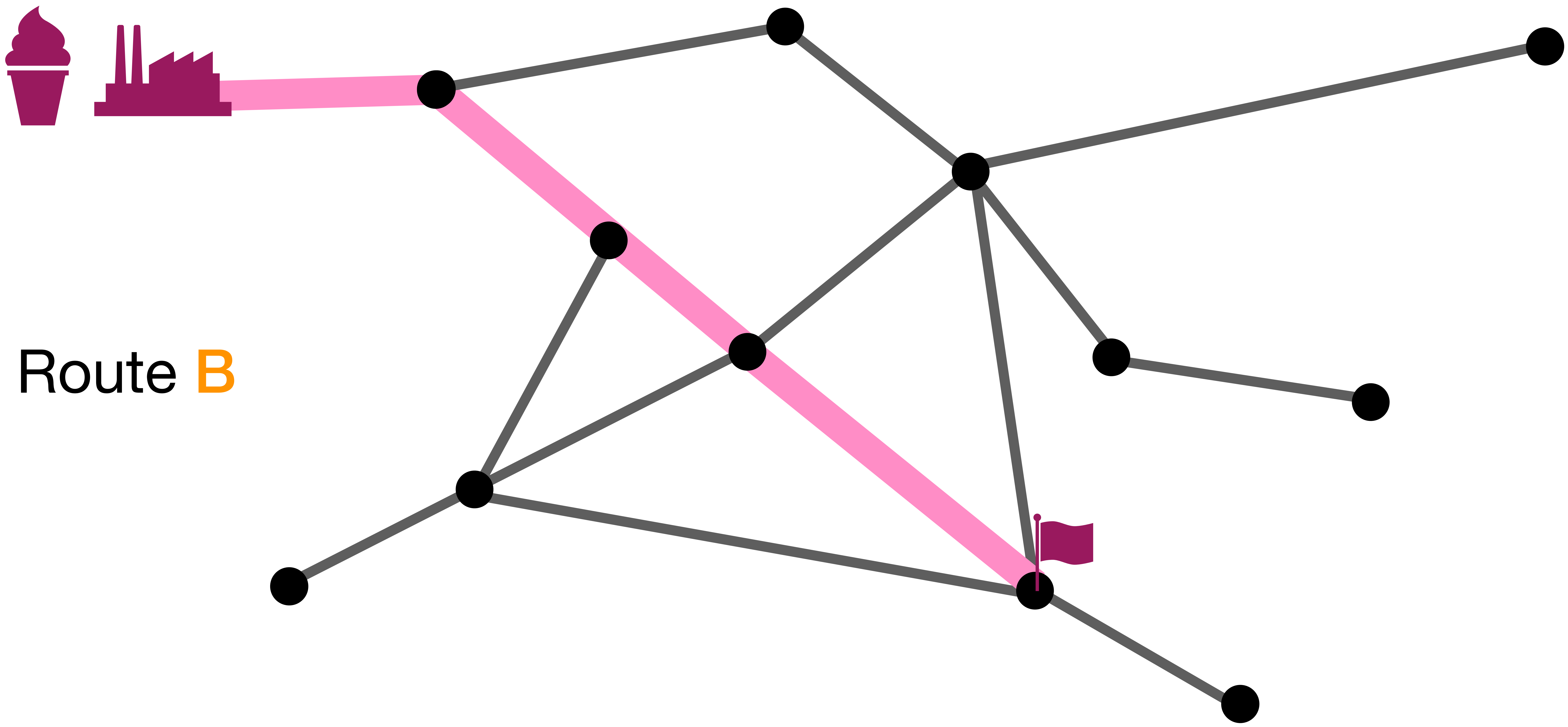
Network routing: introducing the problem



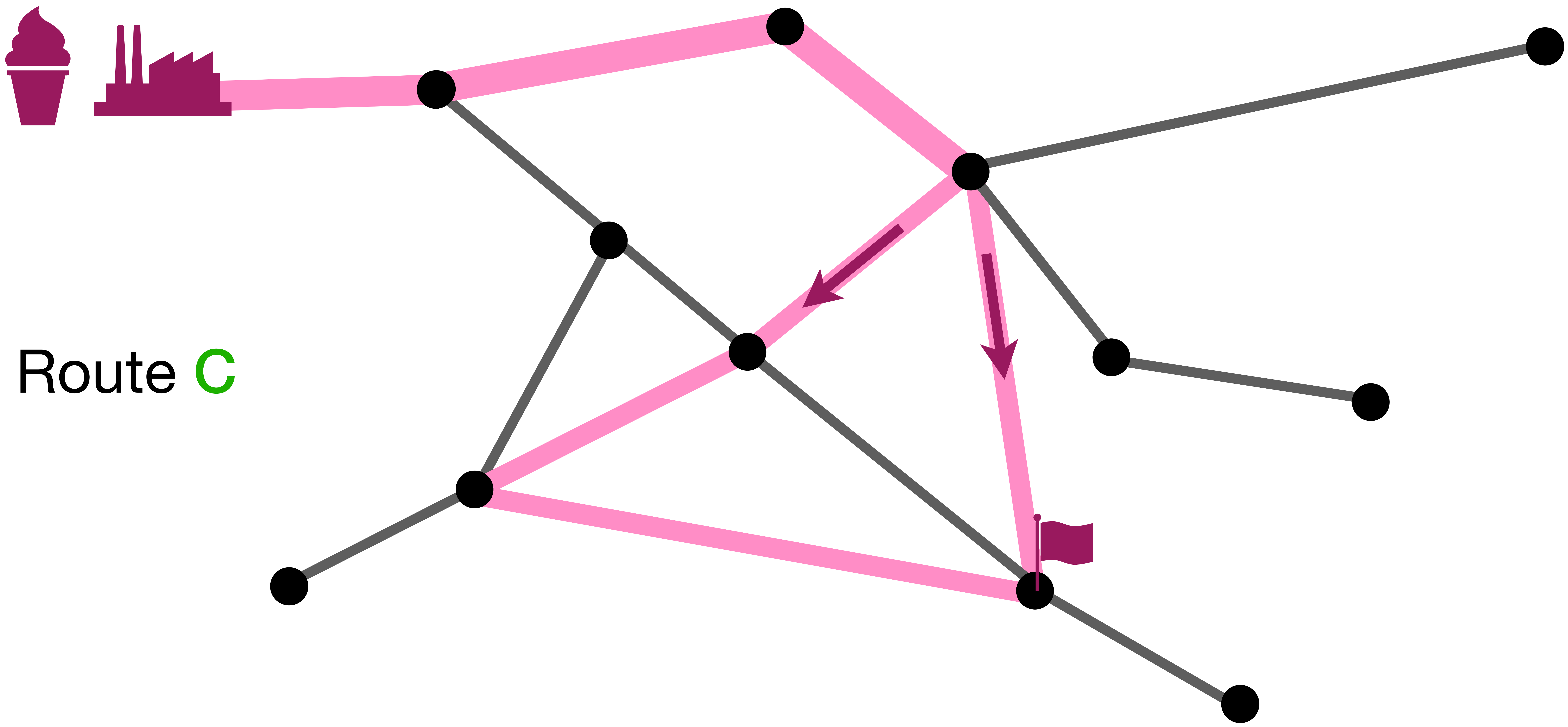
Route A

Network routing: introducing the problem



Route B

Network routing: introducing the problem



Route C

Network routing: formalization

What is the ~~best~~ route?

A B C

Network routing: formalization

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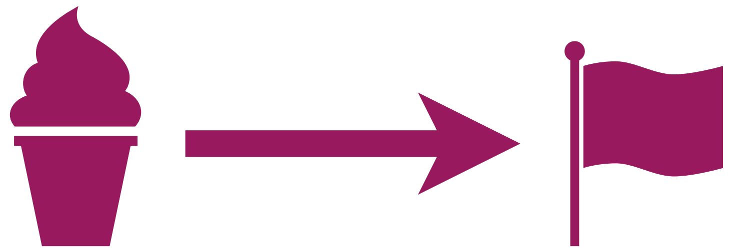
What is the optimal route?

$J($  $)$

g h

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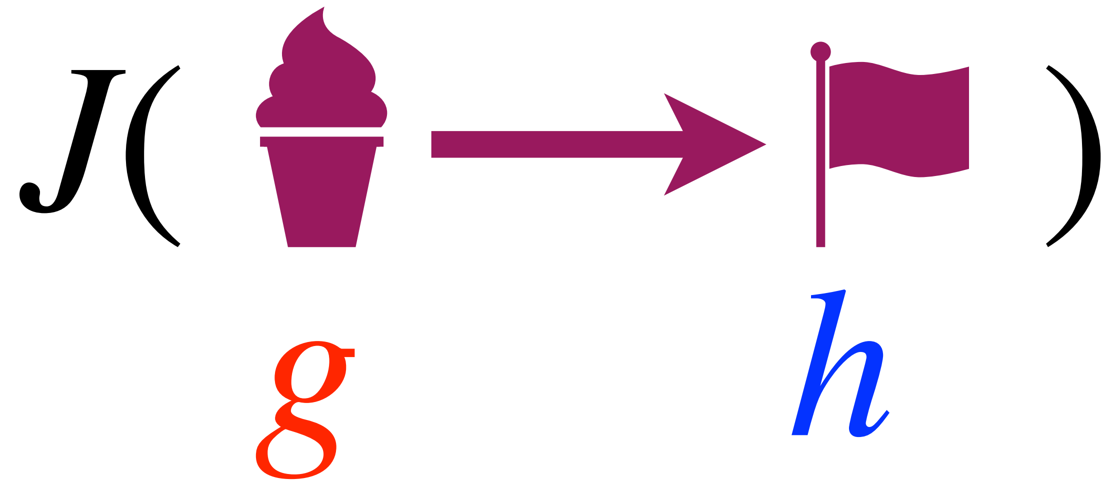

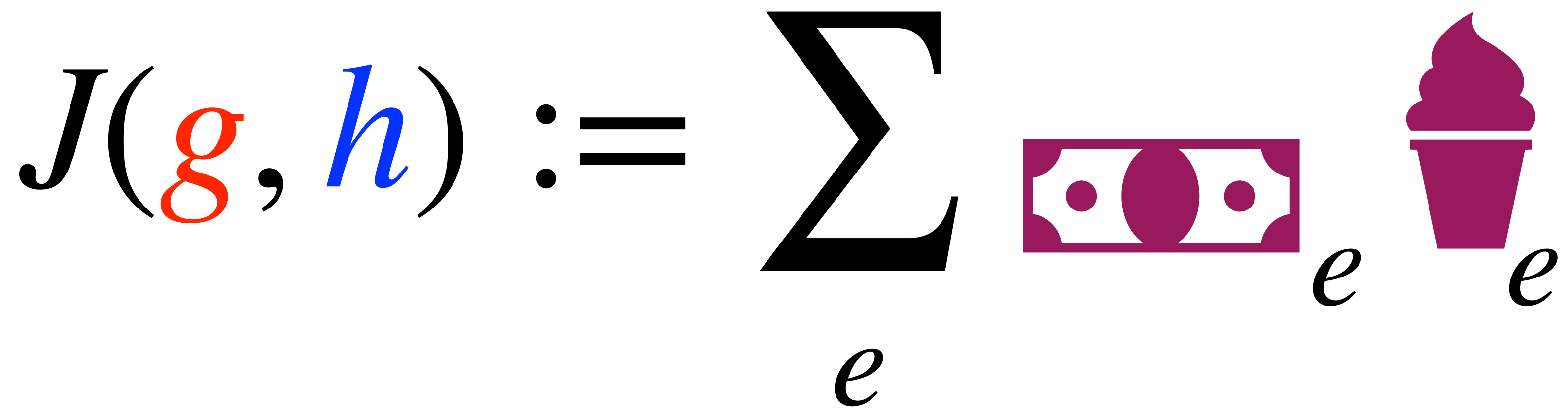

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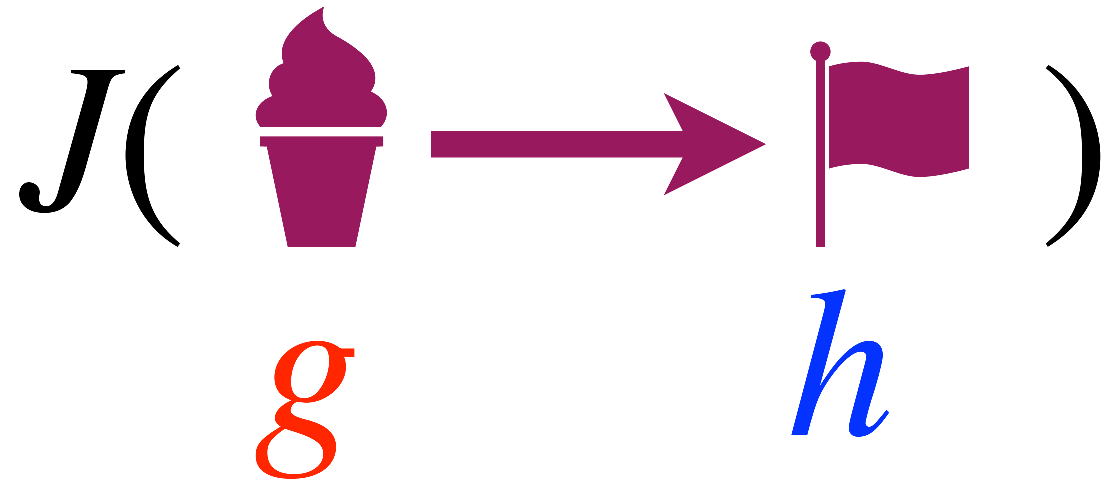

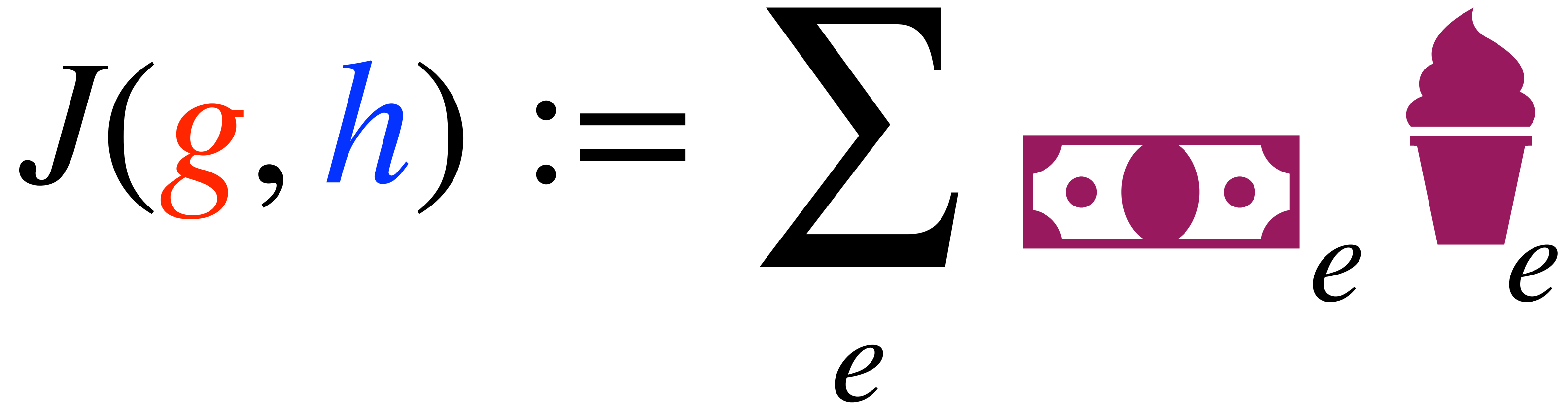

$$J(g,h) := \sum_e w_e |F_e|$$

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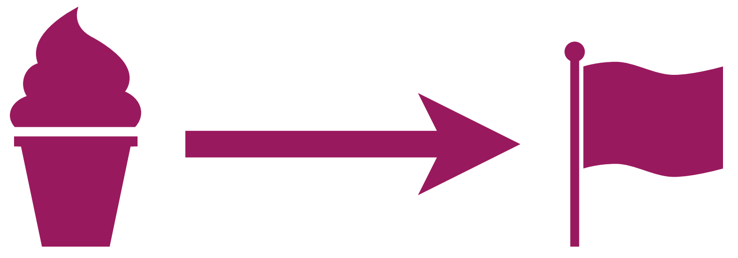
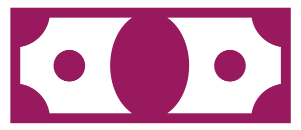
$J($  $)$

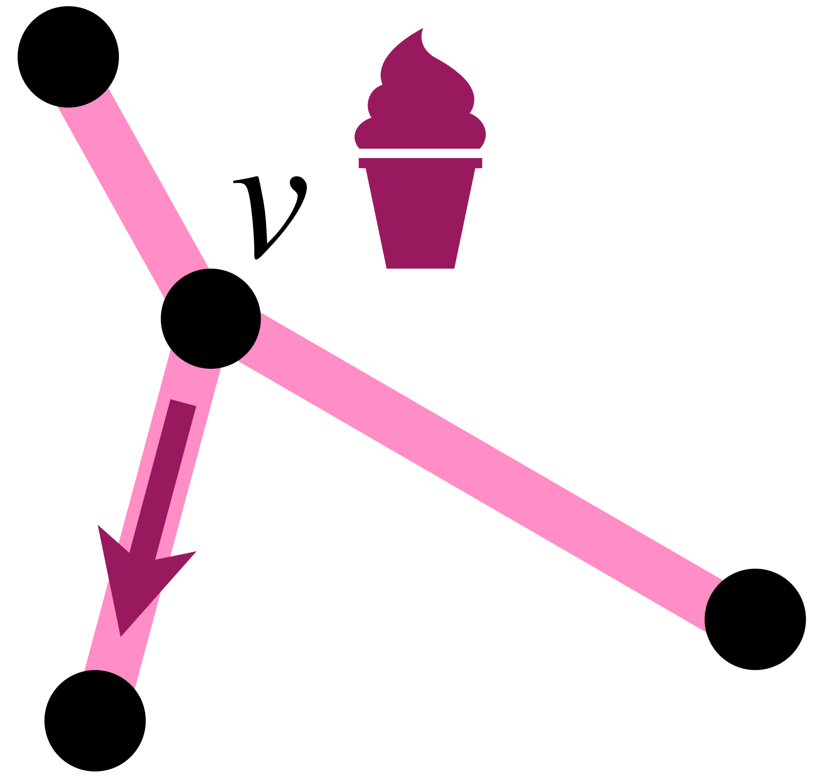
g h

$$J(g, h) := \sum_e$$  $_e$  $_e$

$$J(g, h) := \sum_e w_e |F_e|$$

Conservation constraint

$$g_v - h_v = \sum_{e:e=(v,\cdot)} B_{ve} F_e$$



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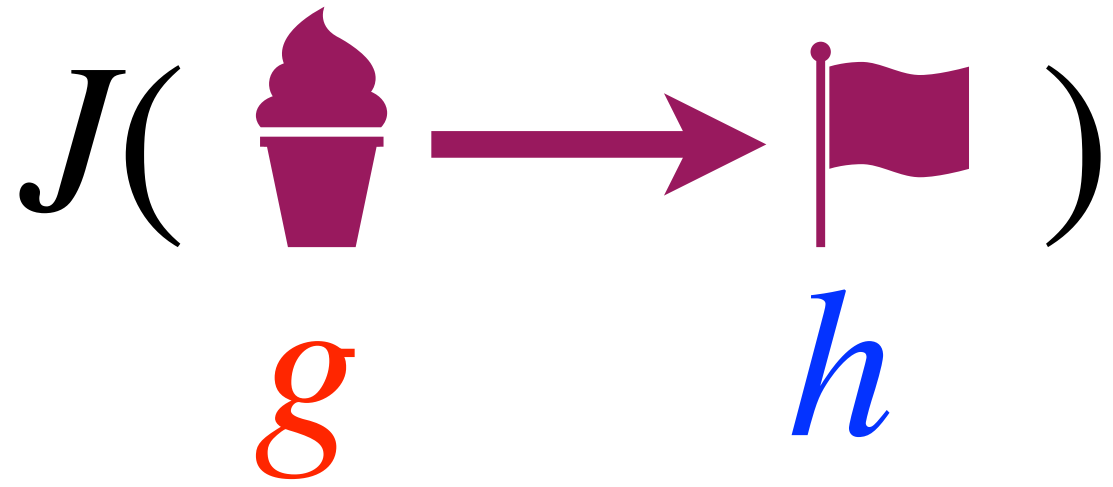

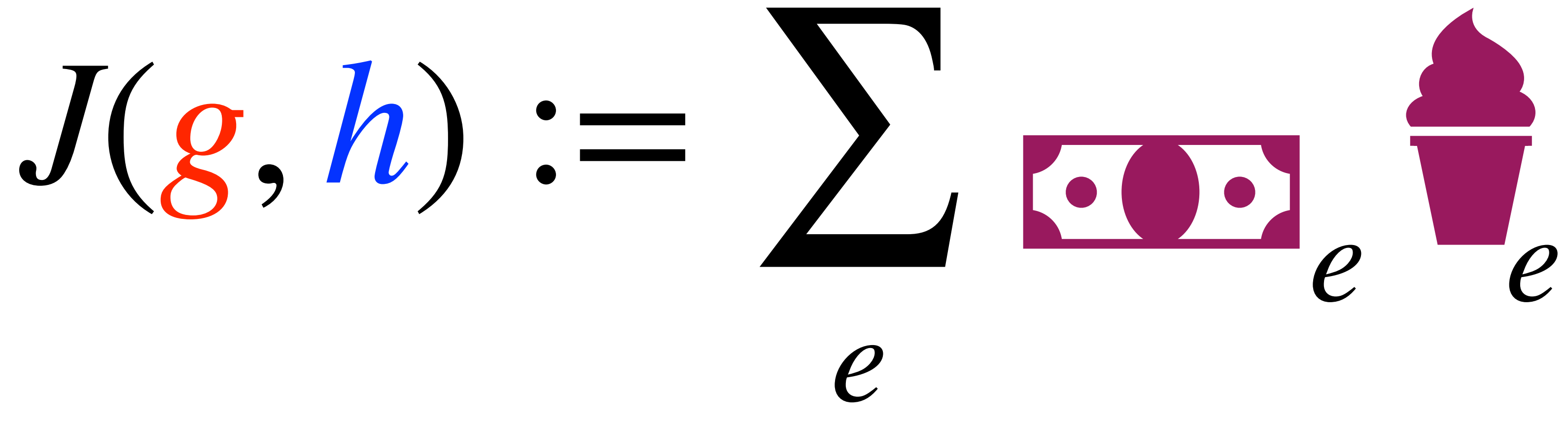

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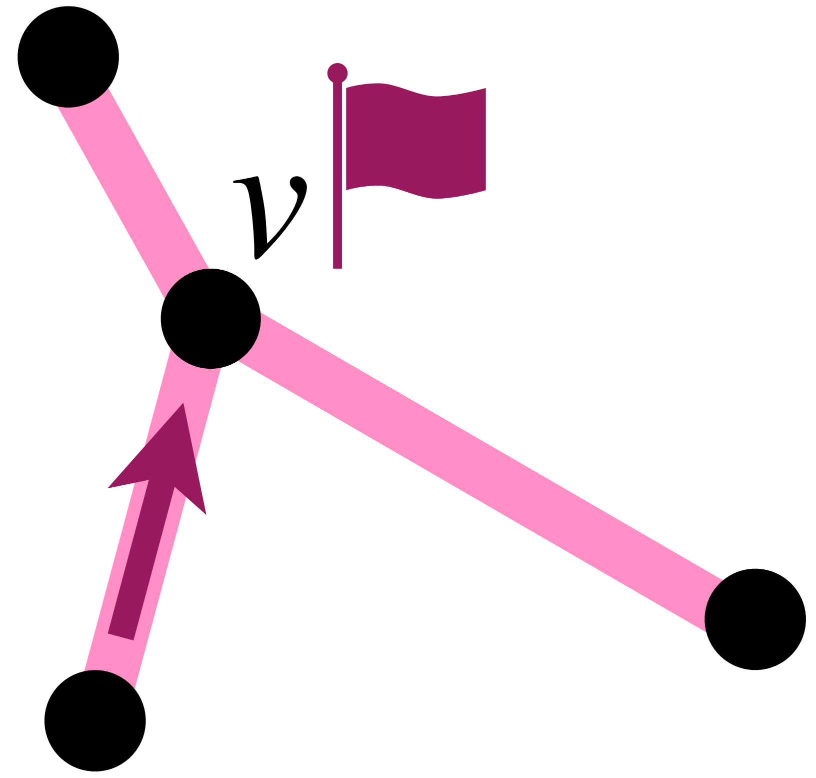

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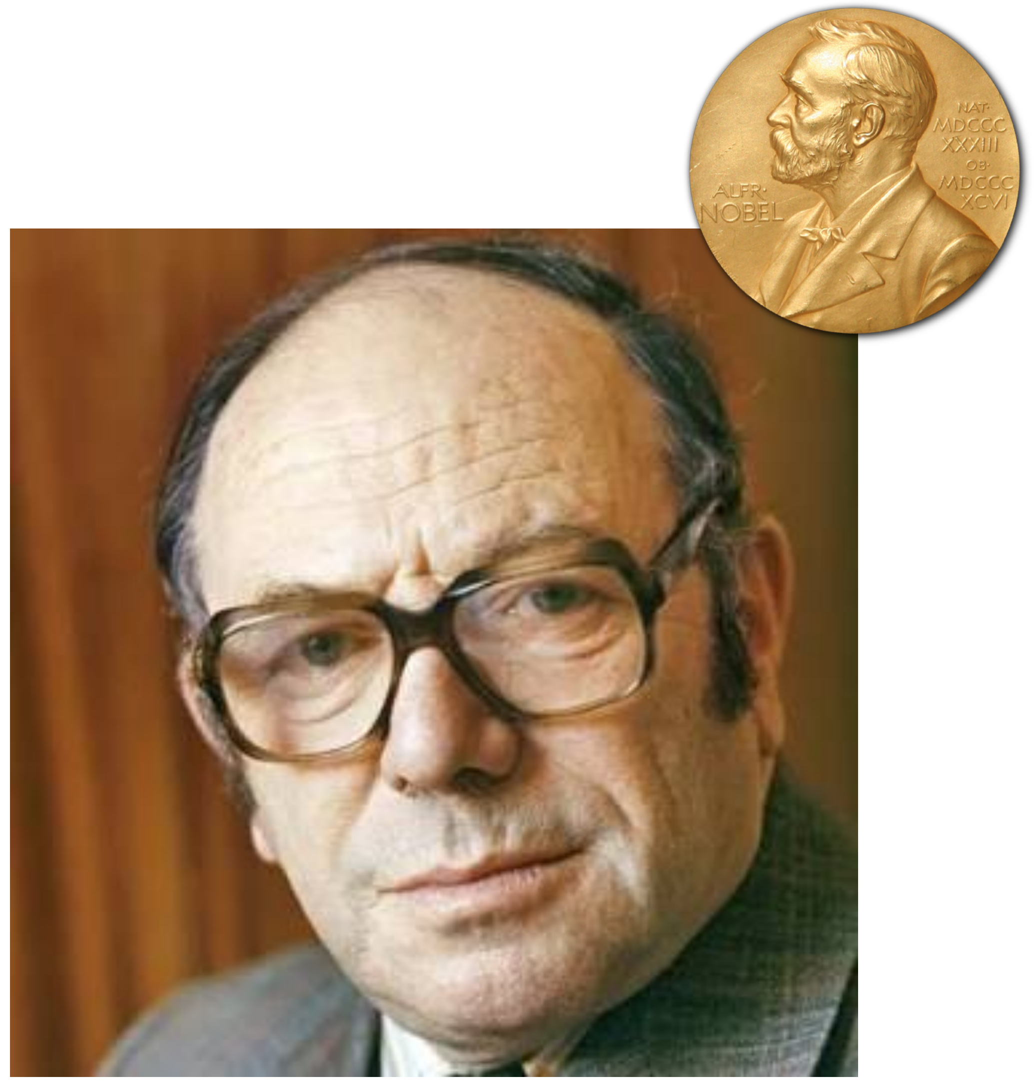
G. Monge
1746 - 1818



L. Kantorovich
1912 - 1986

Optimal transport problem

(and its connection with network routing)

Network routing: formalization

What is the optimal route?

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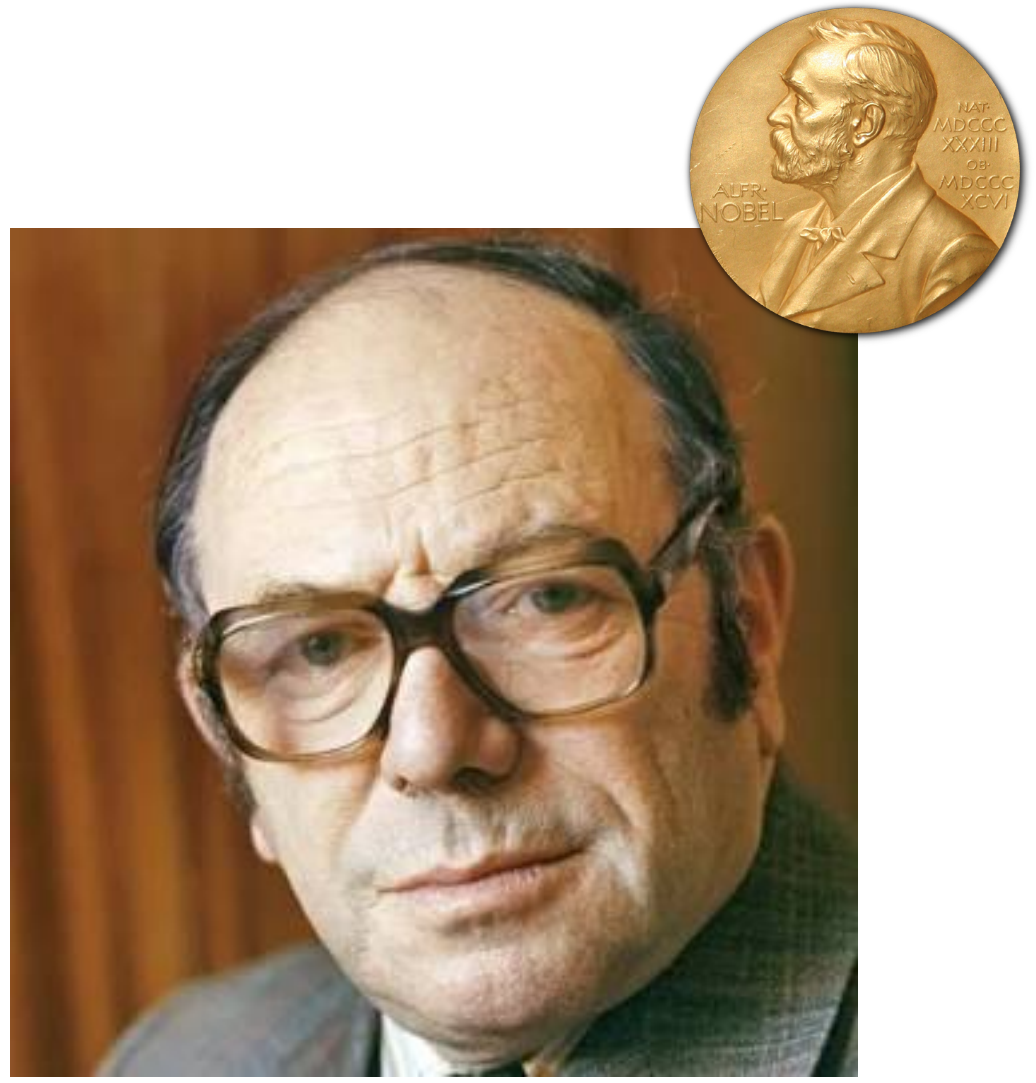
$$W(g,h) := \min_{F \in C(g,h)} J(g,h)$$

(1-) Wasserstein distance

Kantorovich-Rubinstein metric

Earth Mover distance

G. Monge
1746 - 1818



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Optimal transport problem

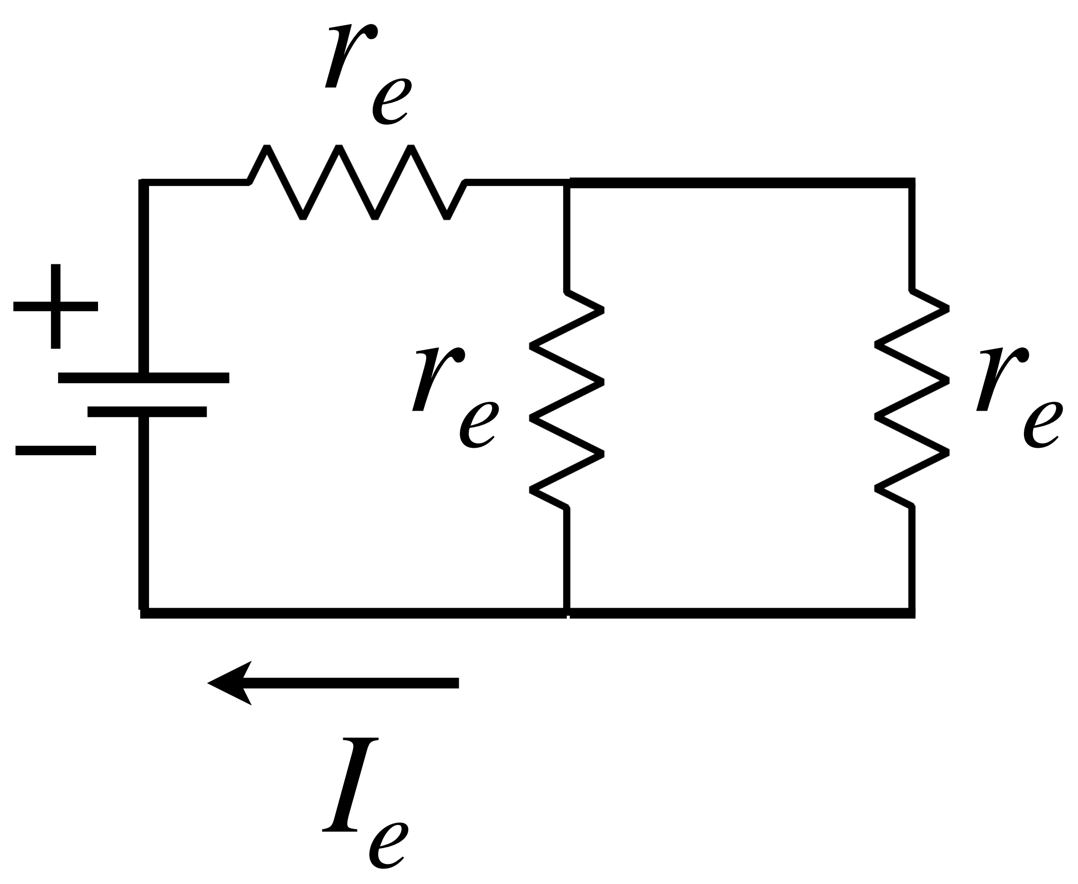
(and its connection with network routing)

Energy minimization

A different take on optimal transport (~'90 - now):

Fittest transportation networks

Optimal channel networks (OCN)



$$V := \sum_e r_e |I_e|$$

$$\min_{I \in \mathcal{J}} V$$

Energy minimization

A different take on optimal transport (~'90 - now):

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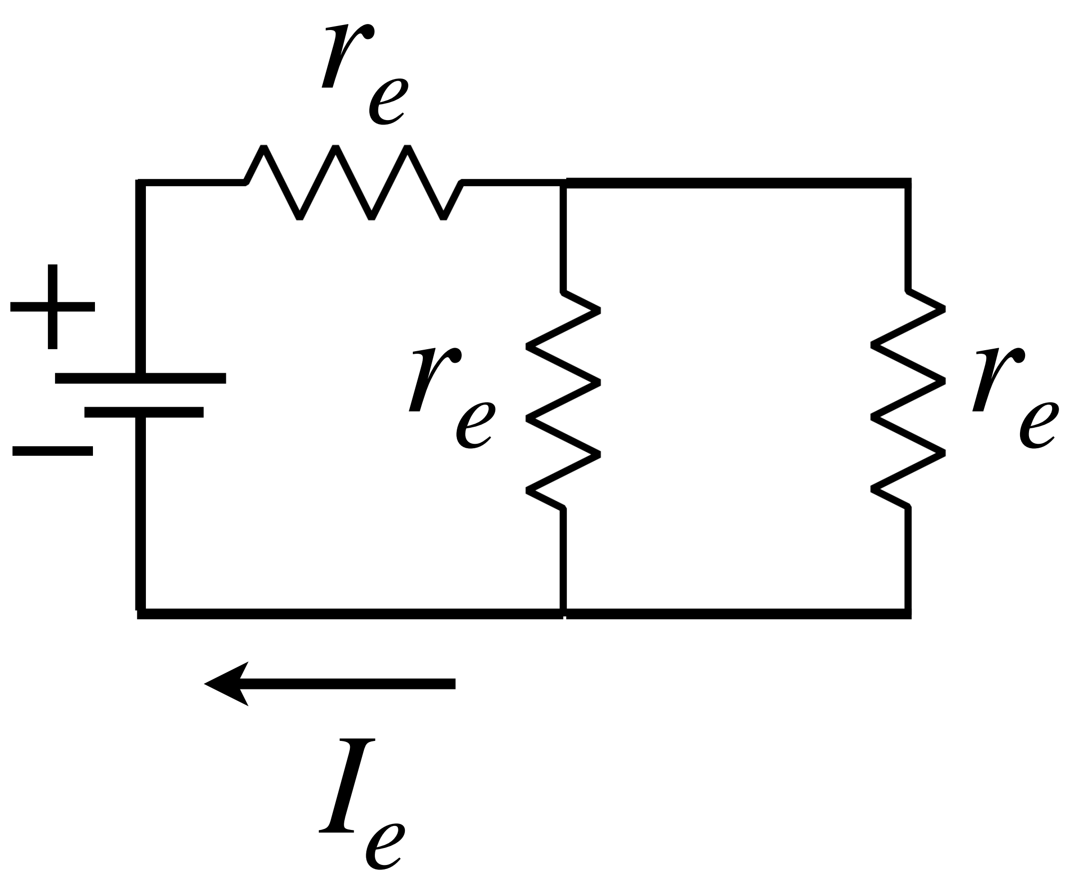
$$V := \sum_e r_e |I_e|^{\gamma}$$

$$\min_{I \in \mathcal{J}} V \quad (0 < \gamma < 2)$$



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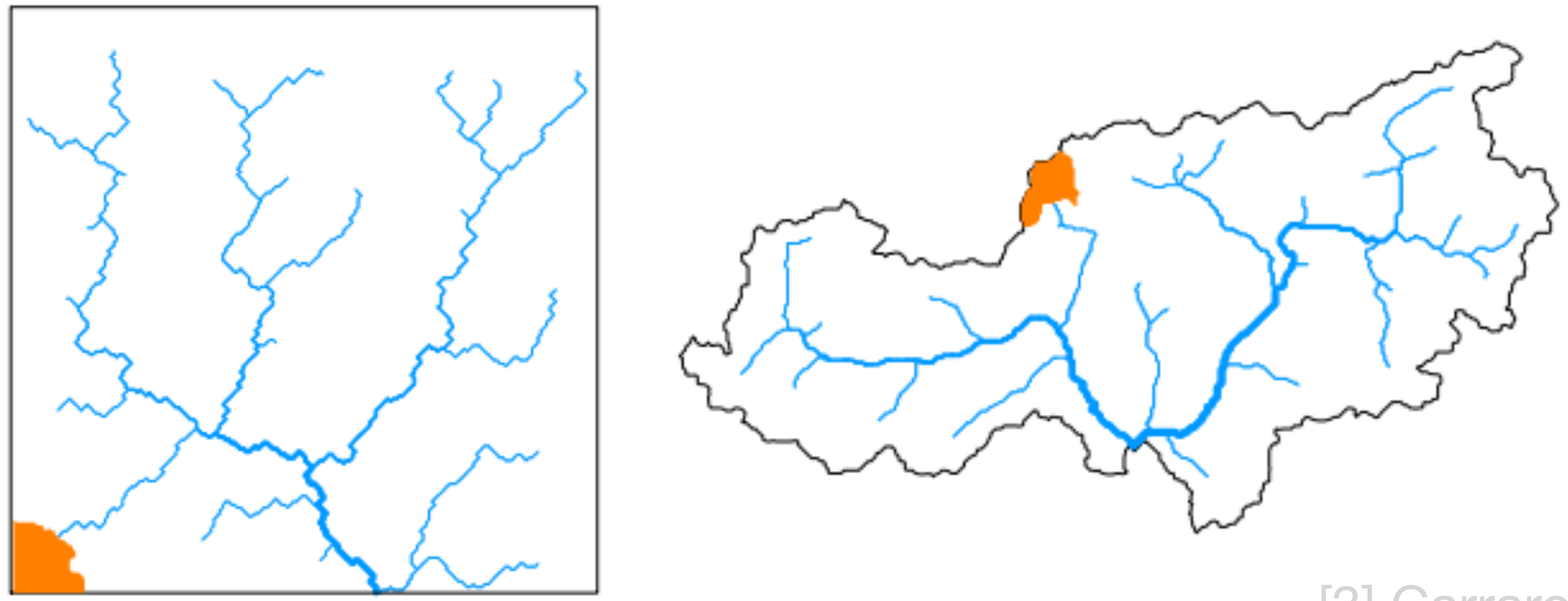

[3] Carraro & Altermatt Nat. Comm. Earth and Env. 2022

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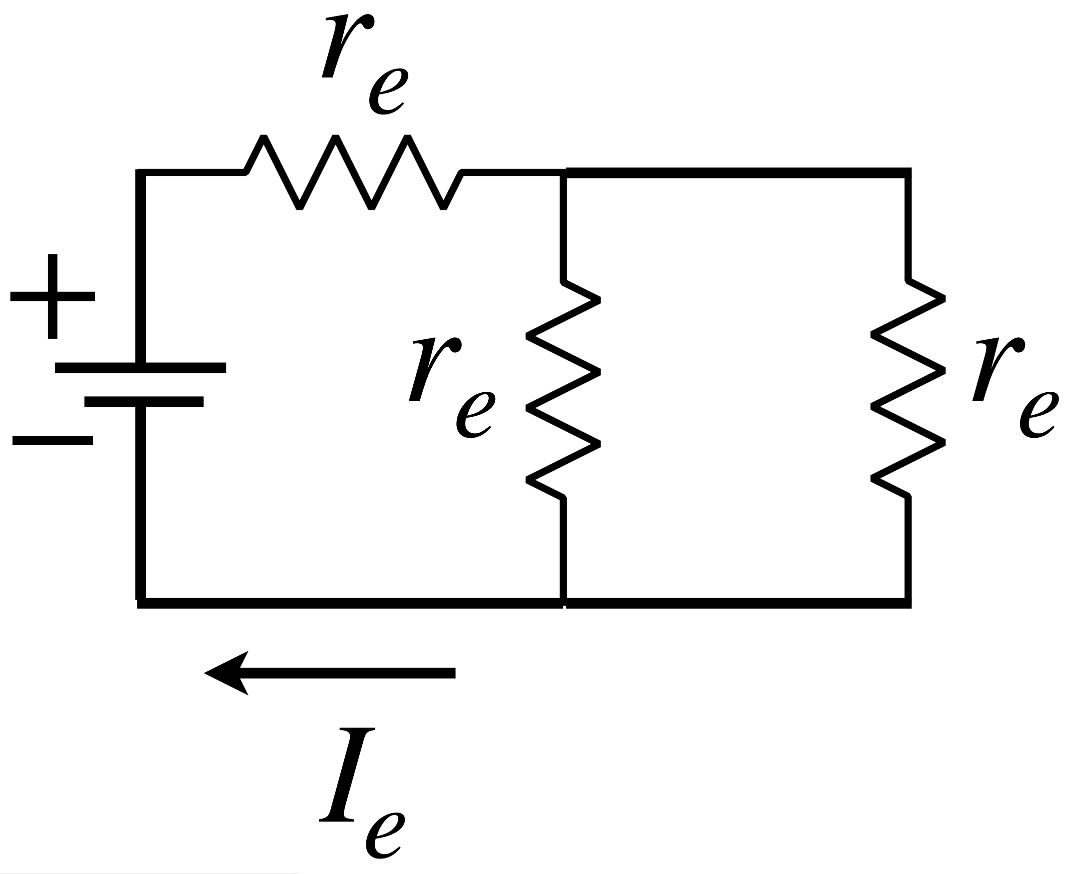

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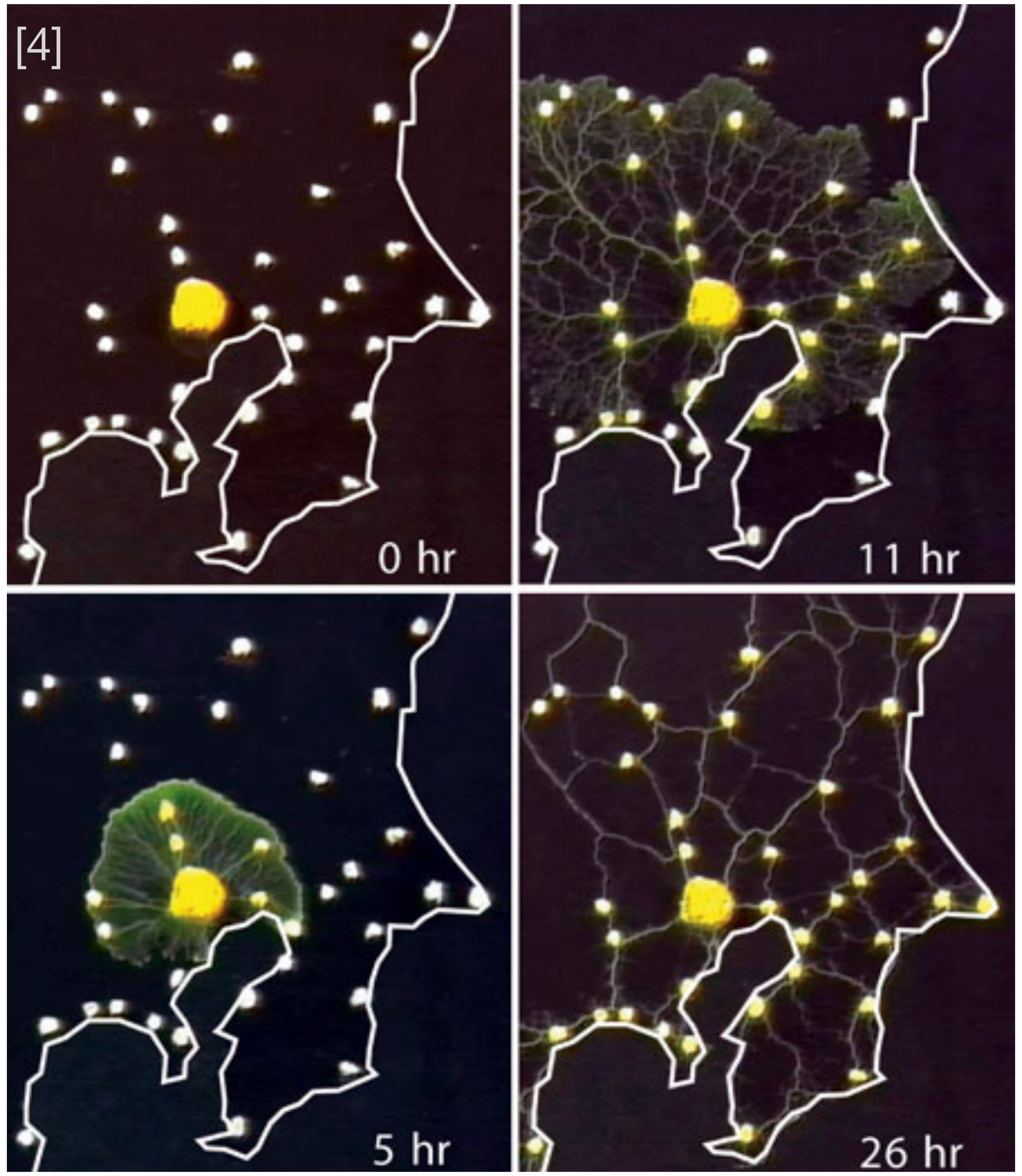
[3]

OT minimization $J := \sum_e w_e |F_e|$

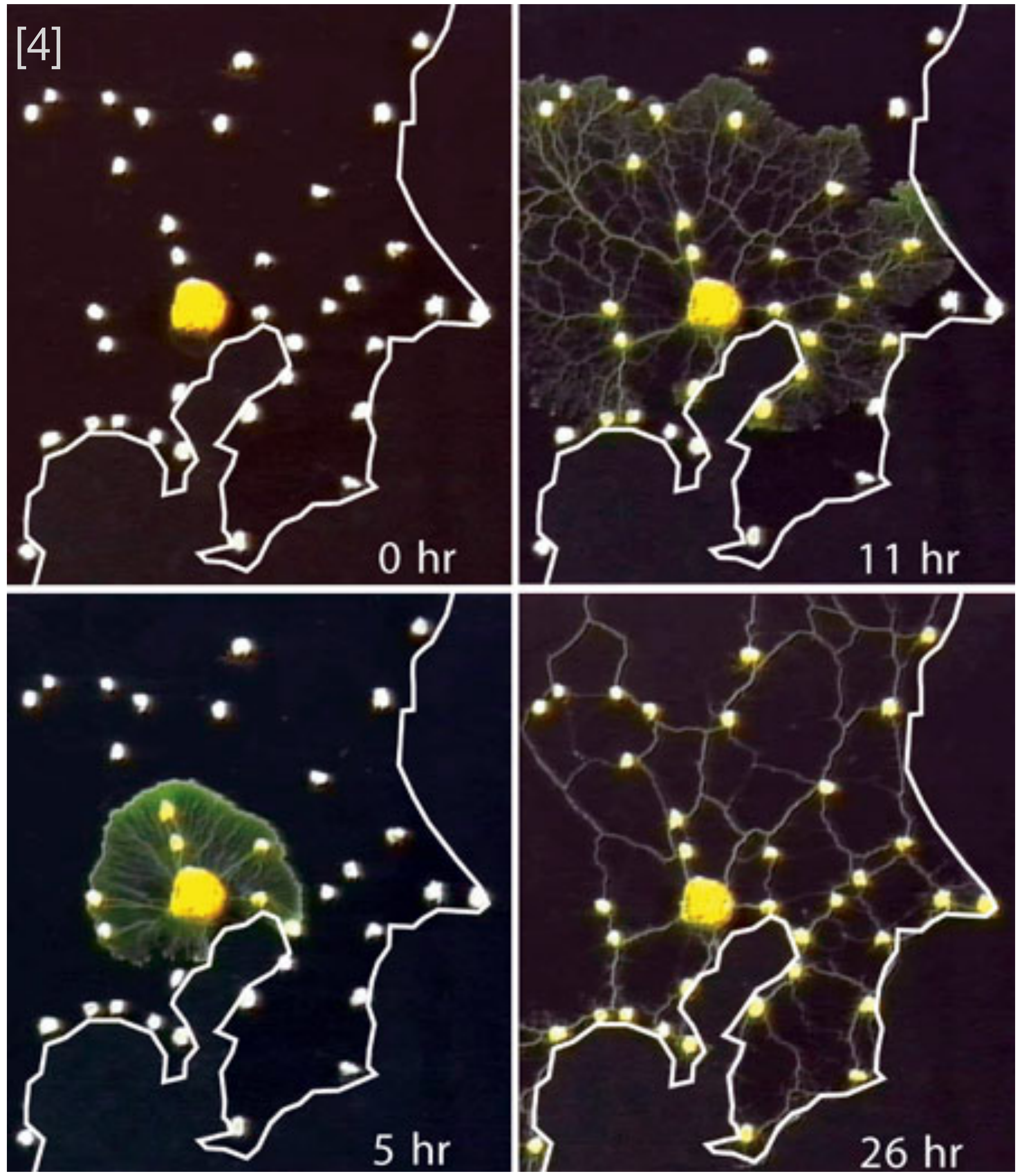
[3] Carraro & Altermatt Nat. Comm. Earth and Env. 2022

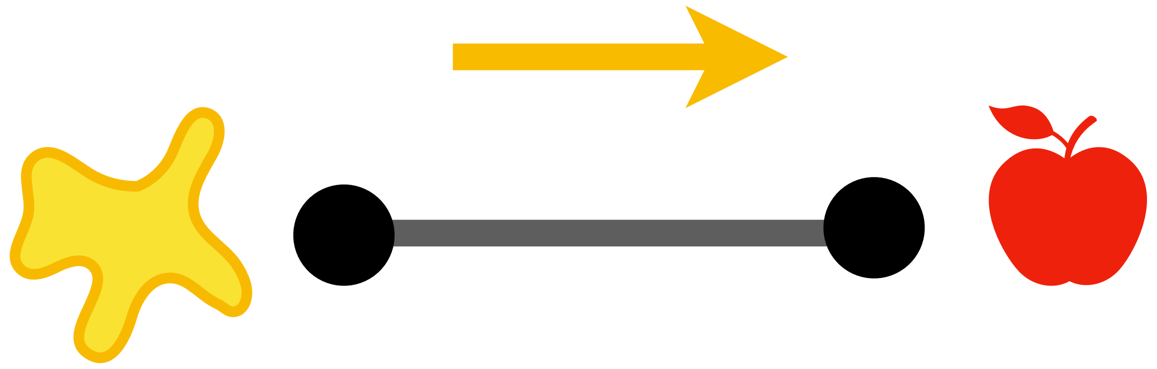
Capacitated networks & adaptation



[4] Tero et al. Science 2010

Capacitated networks & adaptation





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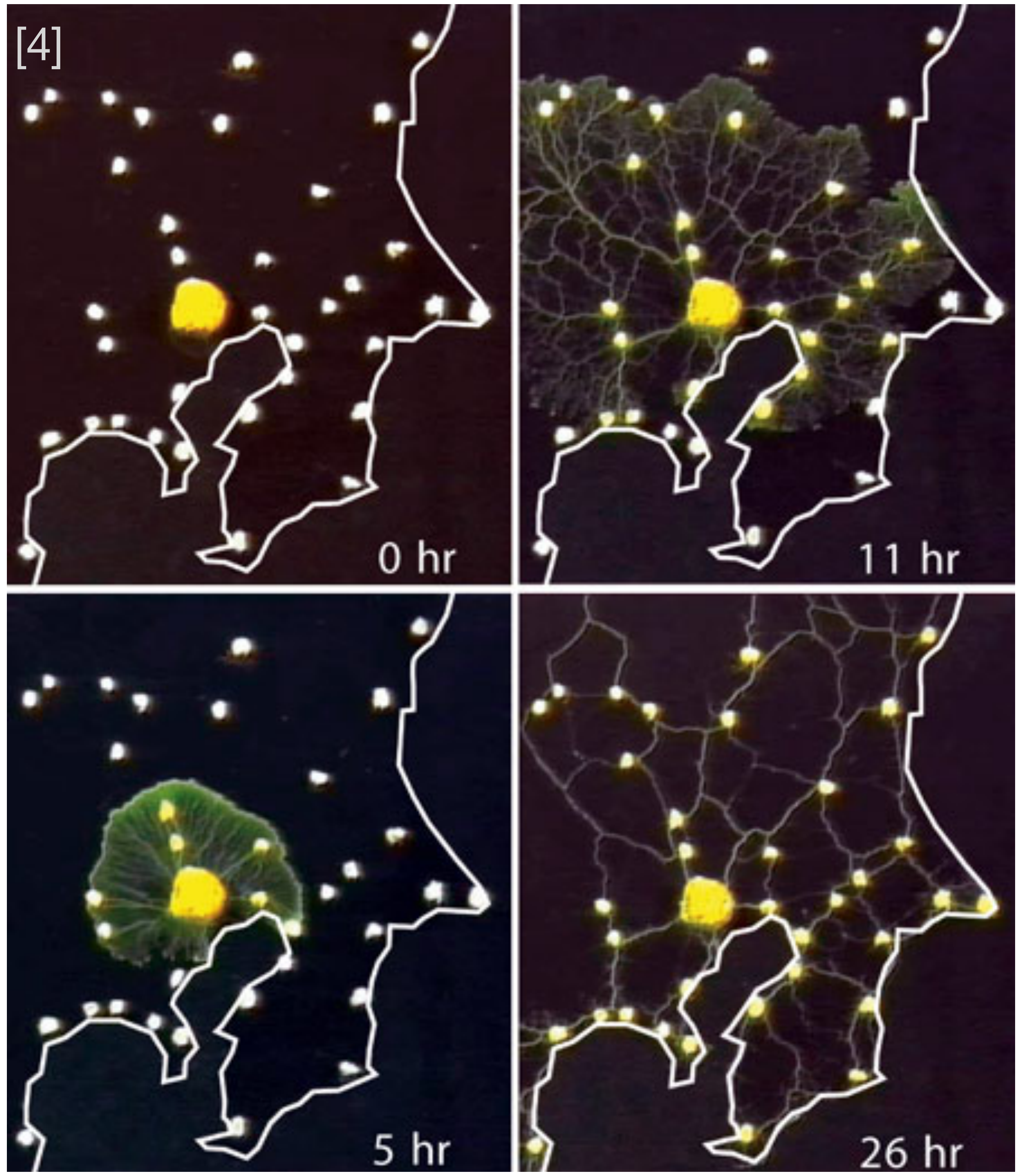

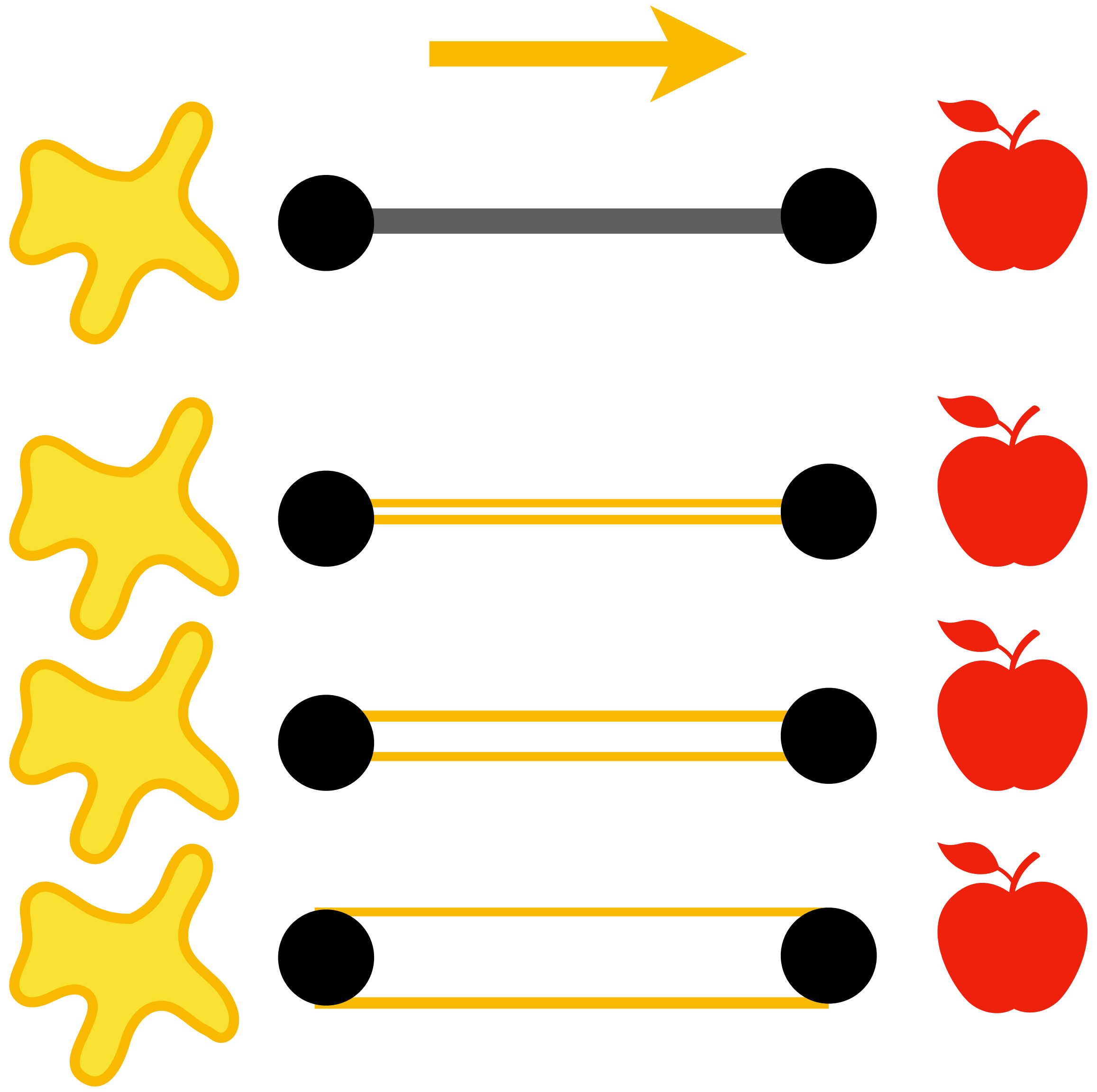

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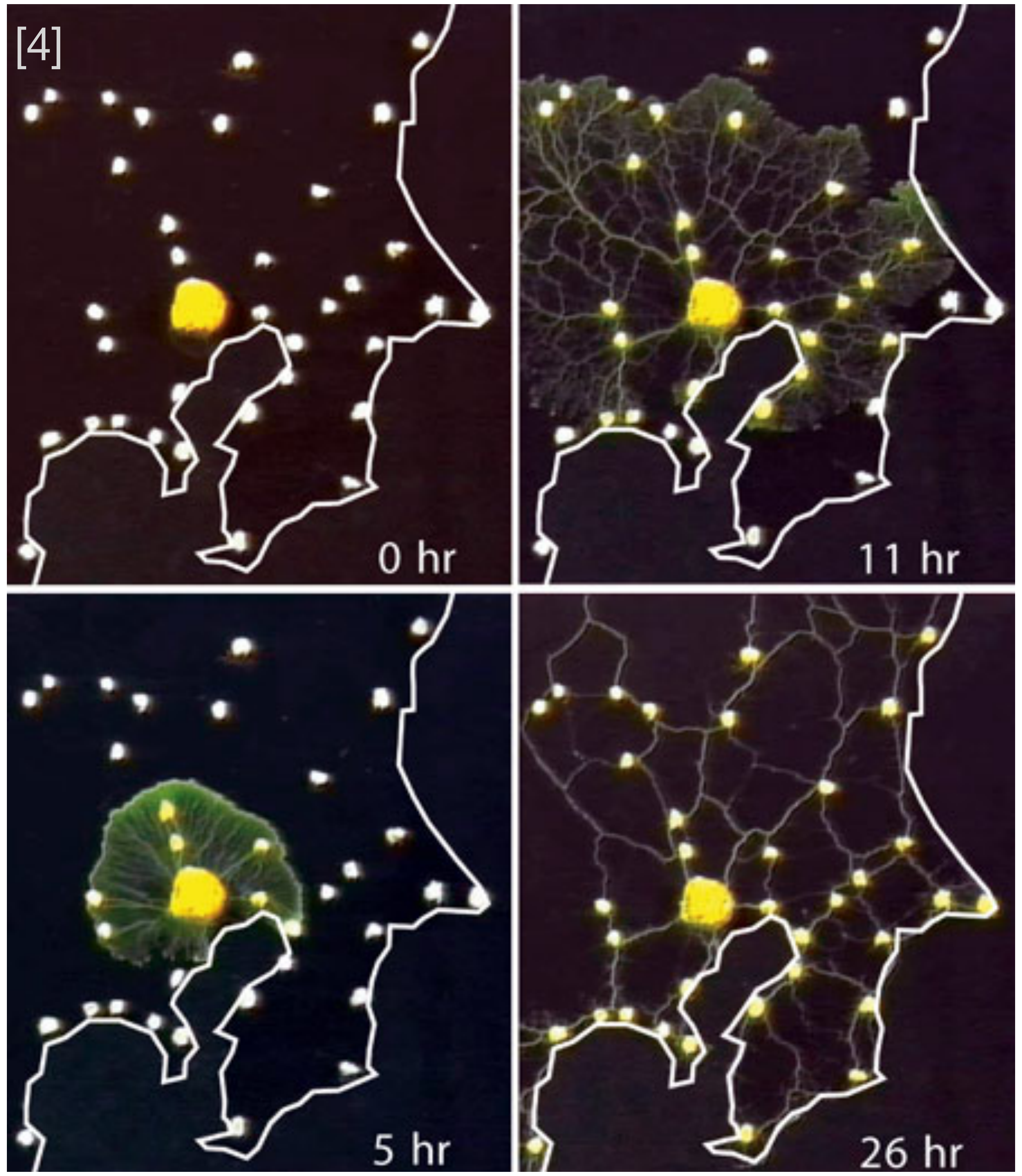

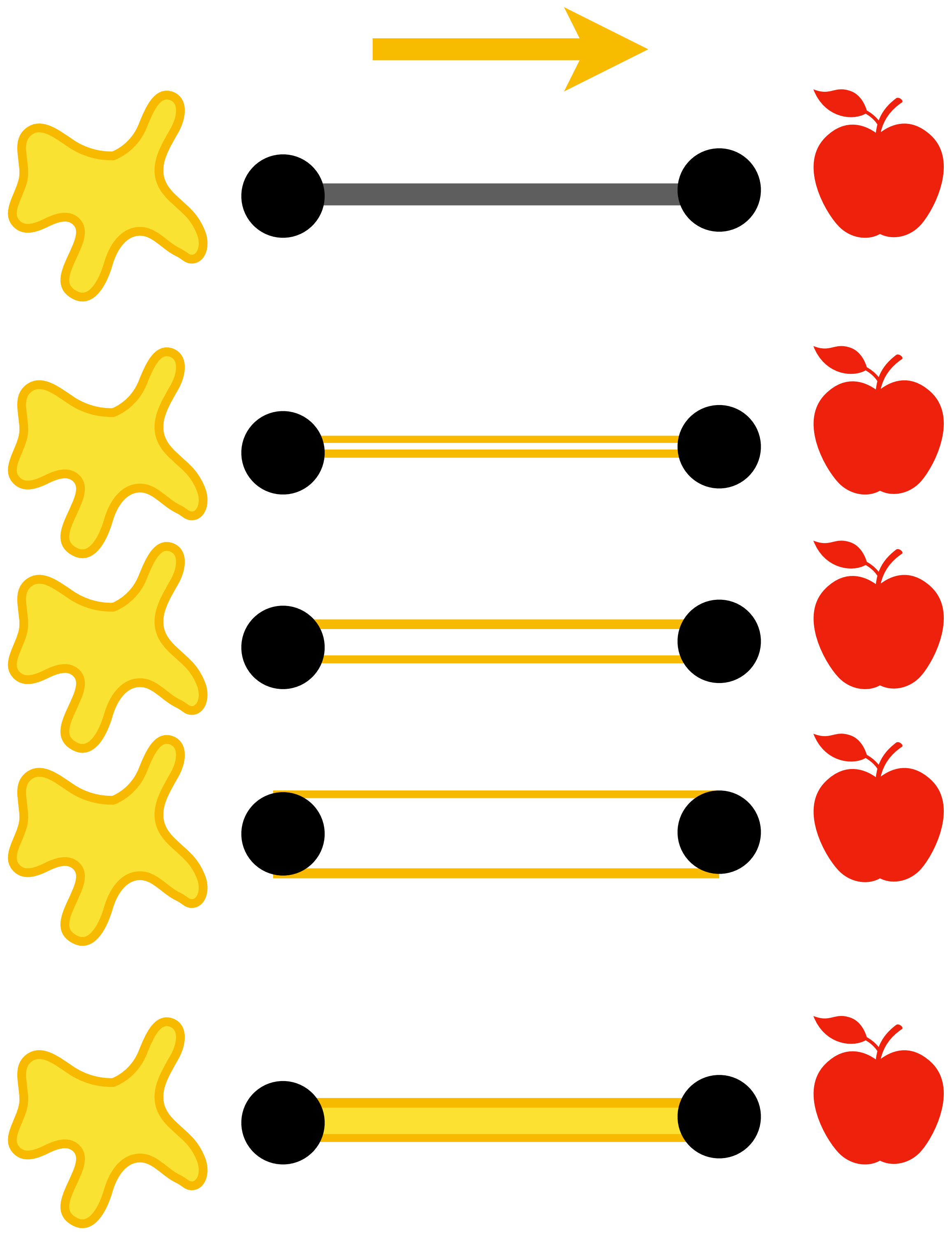

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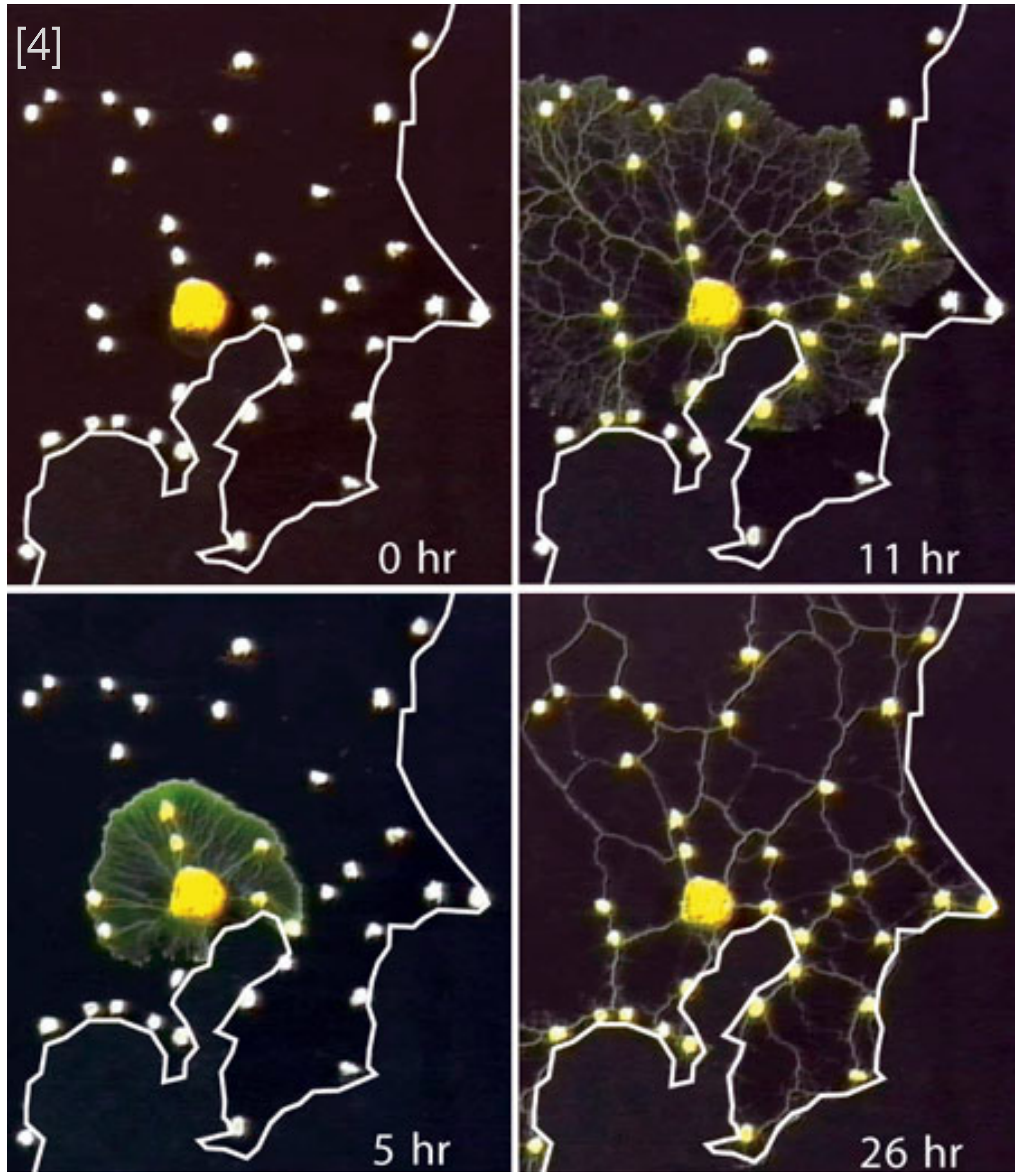
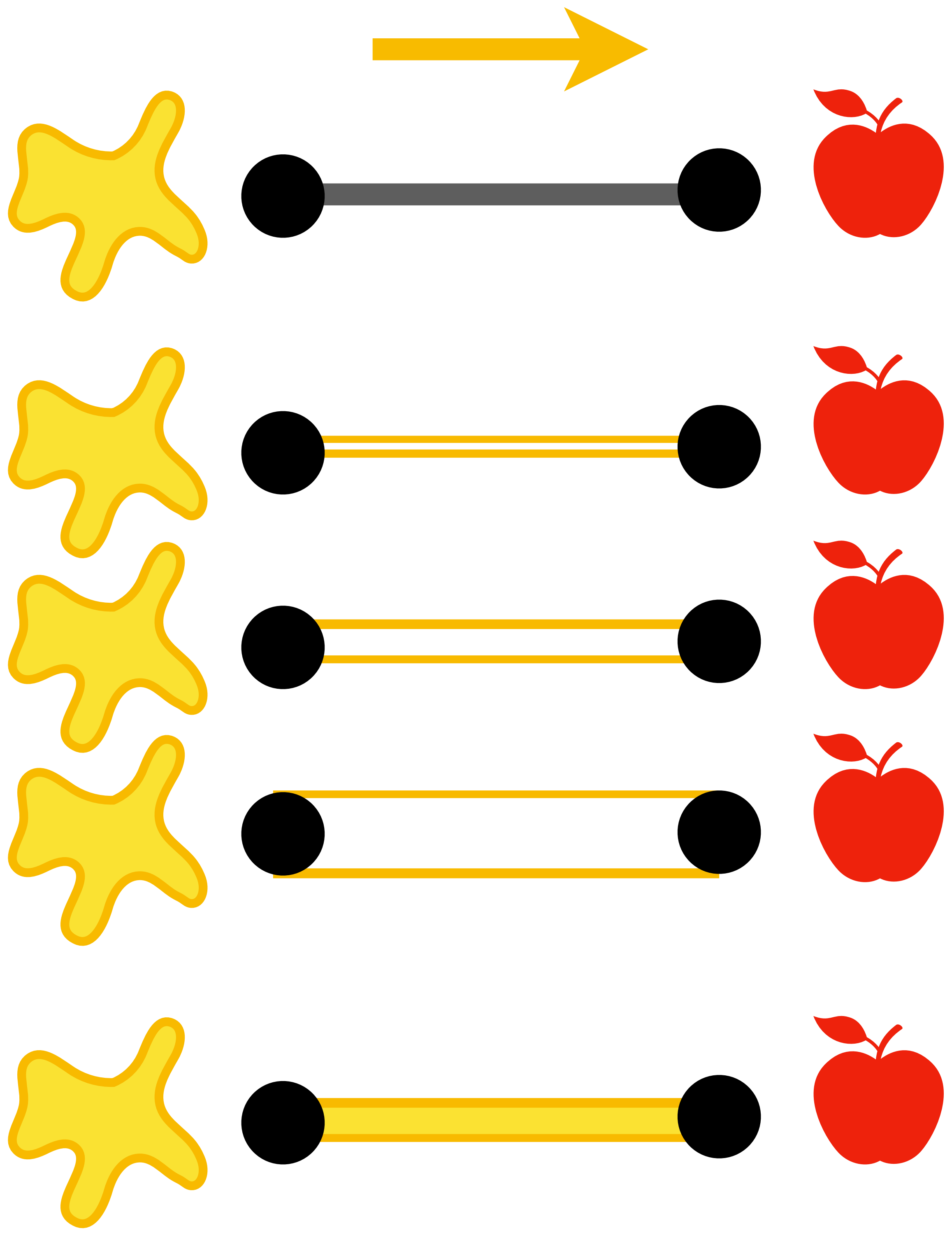
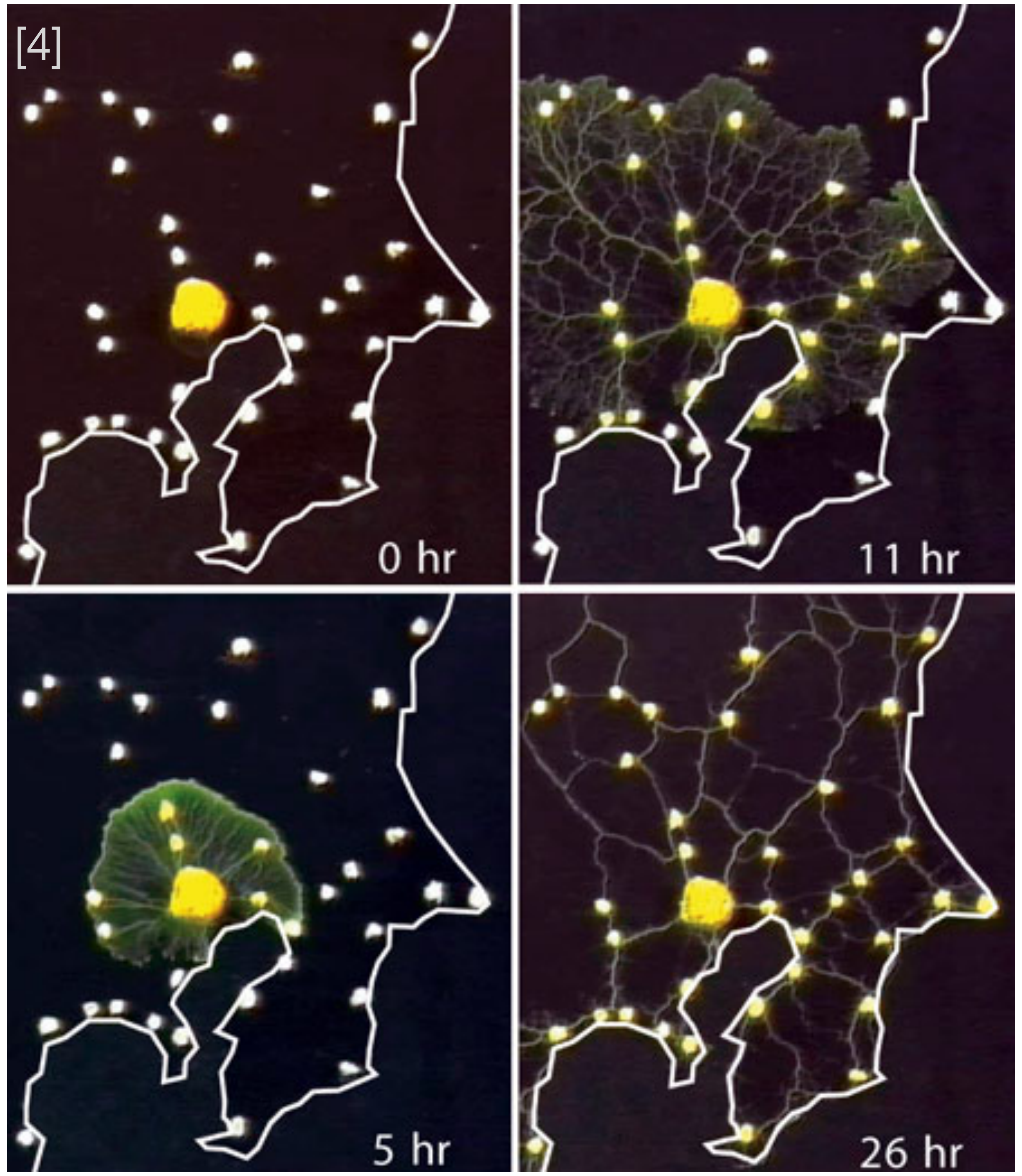


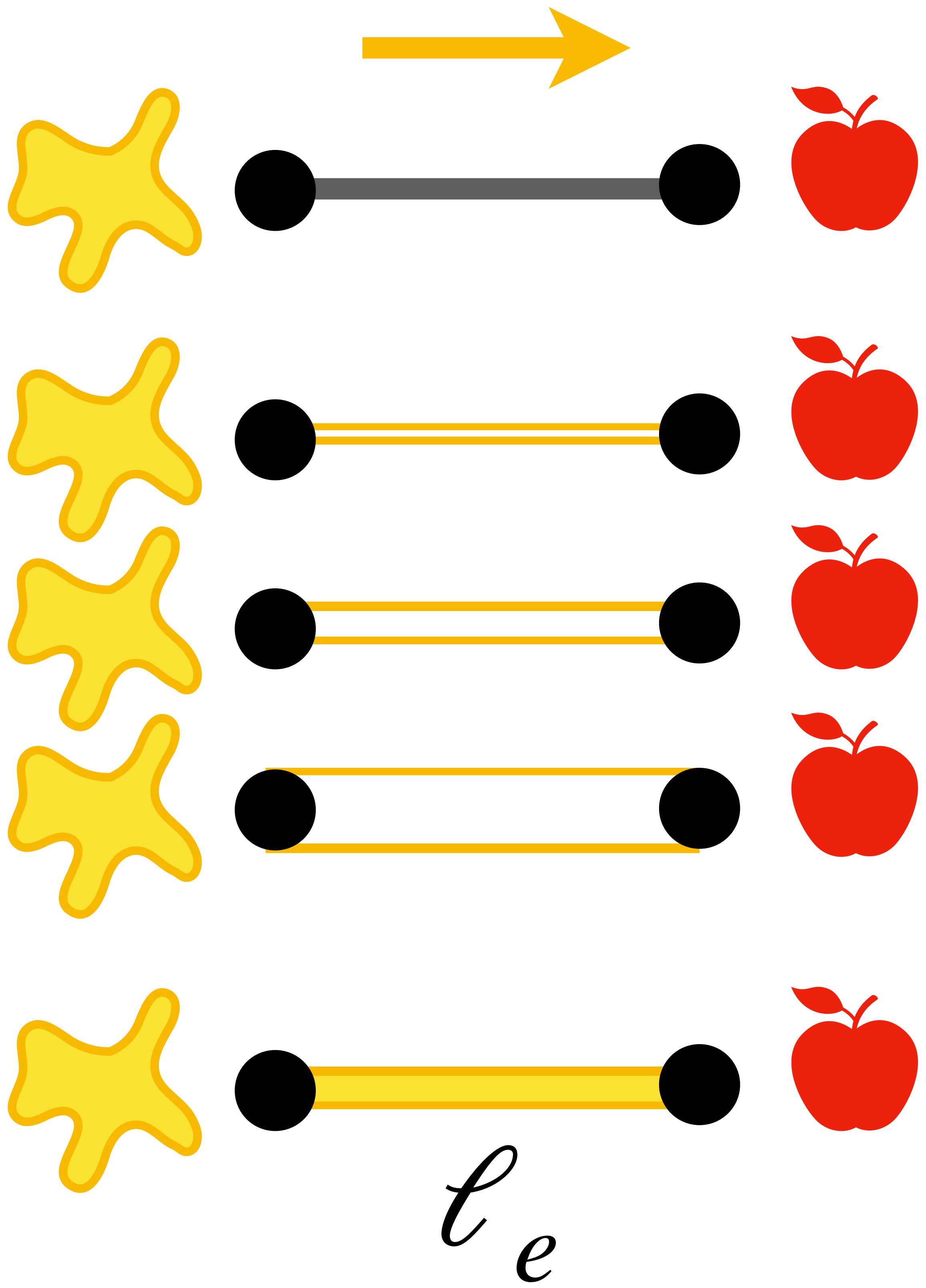


$$\frac{dC_e}{dt} = |F_e| - C_e$$ [5]

[4] Tero et al. Science 2010
[5] Bonifaci et al. J. Theor. Biol. 2012

Capacitated networks & adaptation





Just the right capacity!

[5]

$$\frac{dC_e}{dt} = |F_e| - C_e$$

$$\boxed{E := \sum_e \ell_e |F_e(C_e)|}$$

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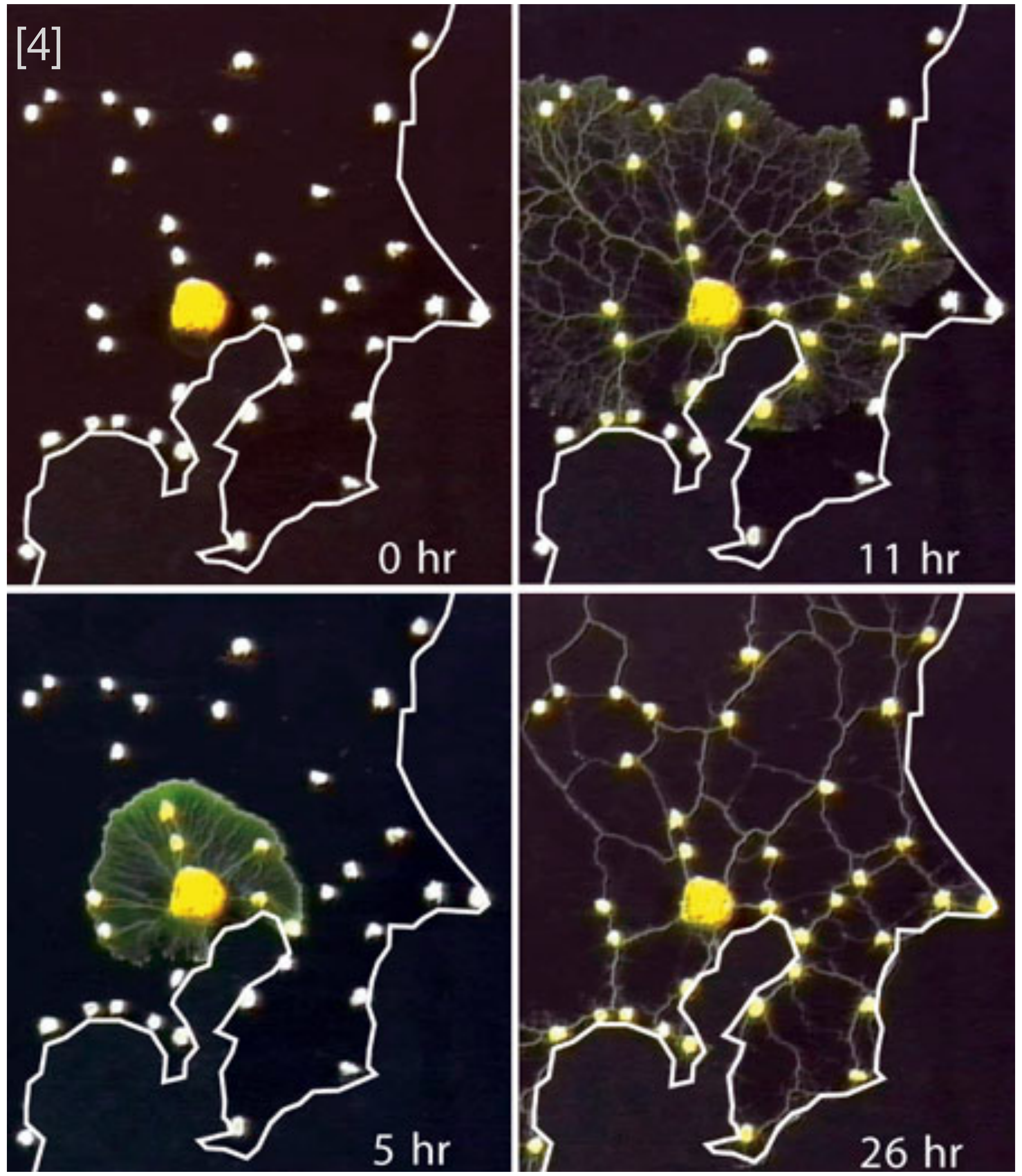

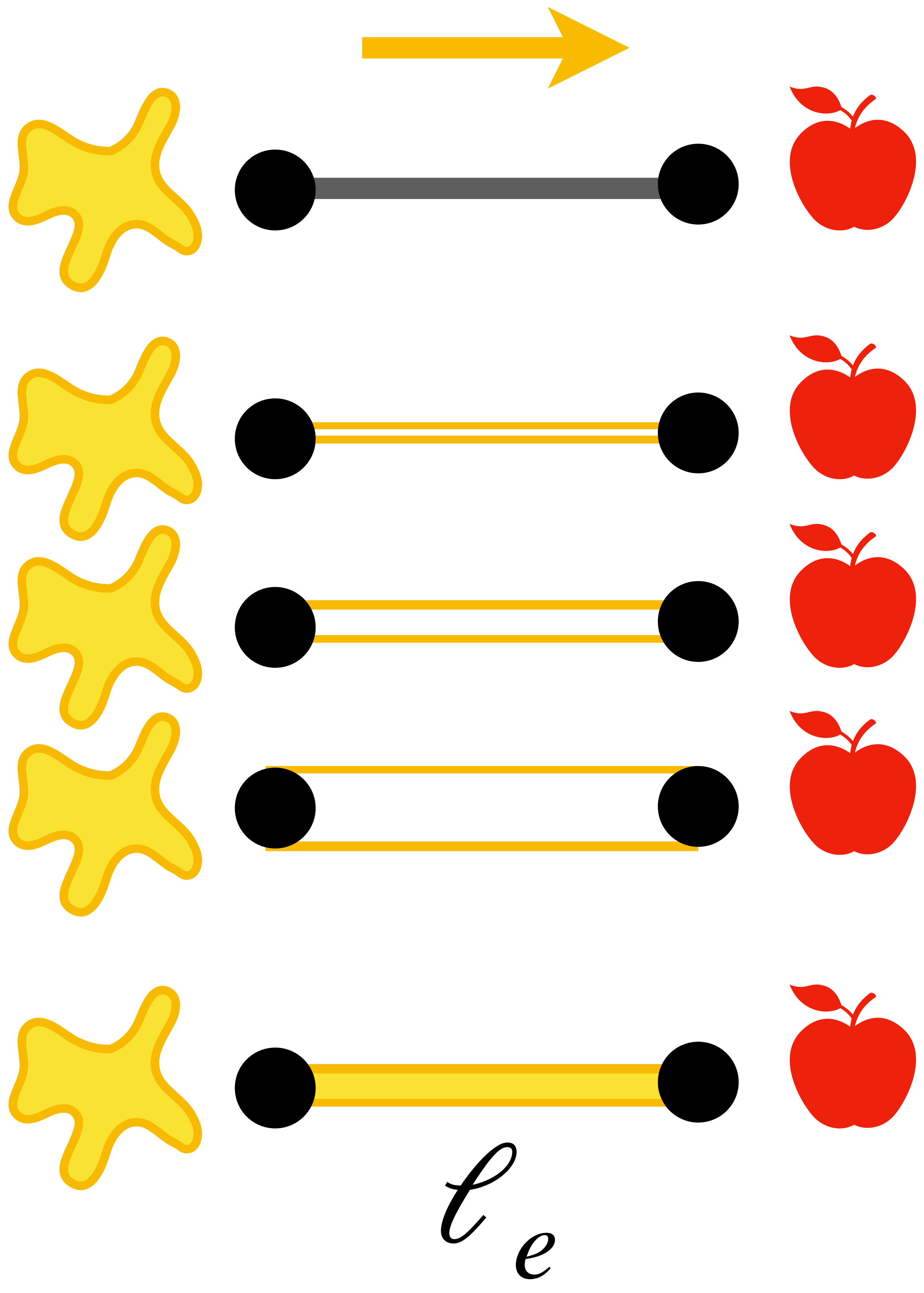

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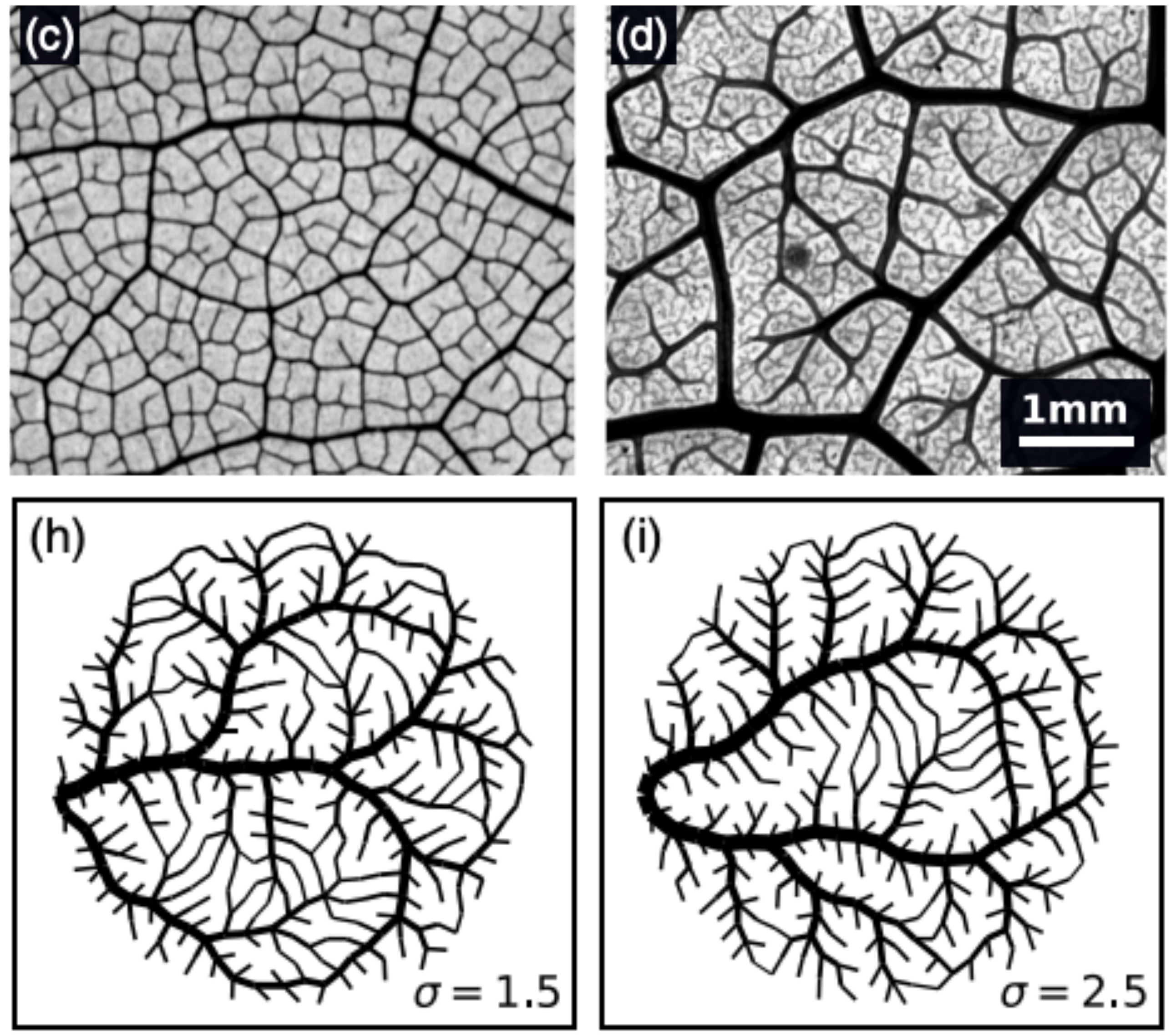

[6]

[4] Tero et al. Science 2010
[5] Bonifaci et al. J. Theor. Biol. 2012
[6] Ronellenfitsch & Katifori PRL 2019

Optimal transport, energy minimization, adaptation

1) $W(g, h) := \min_{F \in C(g,h)} J(g, h)$

$$J(g, h) := \sum_e w_e |F_e|$$

2) $\min_{I \in \mathcal{J}} V$

$$V := \sum_e r_e |I_e|^{\gamma}$$

3) $\dfrac{dC_e}{dt} = |F_e| - C_e \quad \xrightarrow{\min} \quad J := \sum_e \ell_e |F_e(C_e)|$

Optimal transport, energy minimization, adaptation

1) ∪ 2) ∪ 3)

$$\frac{dC_e}{dt} = C_e^{\beta-1}|F_e| - C_e$$

$$J := \sum_e \ell_e |F_e(C_e)|^{\gamma(\beta)}$$

$0 < \beta < 2$

$\beta \geq 1 \rightarrow 0 < \gamma \leq 1$

$0 < \beta < 1 \rightarrow \gamma > 1$

Optimal transport, energy minimization, adaptation

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Computationally efficient method

$0 < \beta < 2$

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Gap in knowledge

Interaction of multiple commodities sharing a unique transportation network [7]

[7] Lonardi et al. PRR 2021

Gap in knowledge

Interaction of multiple commodities sharing a unique transportation network [7]

$$F_e \to \overrightarrow{F_e} = (F_e^1, \ldots, F_e^i, \ldots, F_e^M) \qquad C_e^i := \hat{C}_e$$

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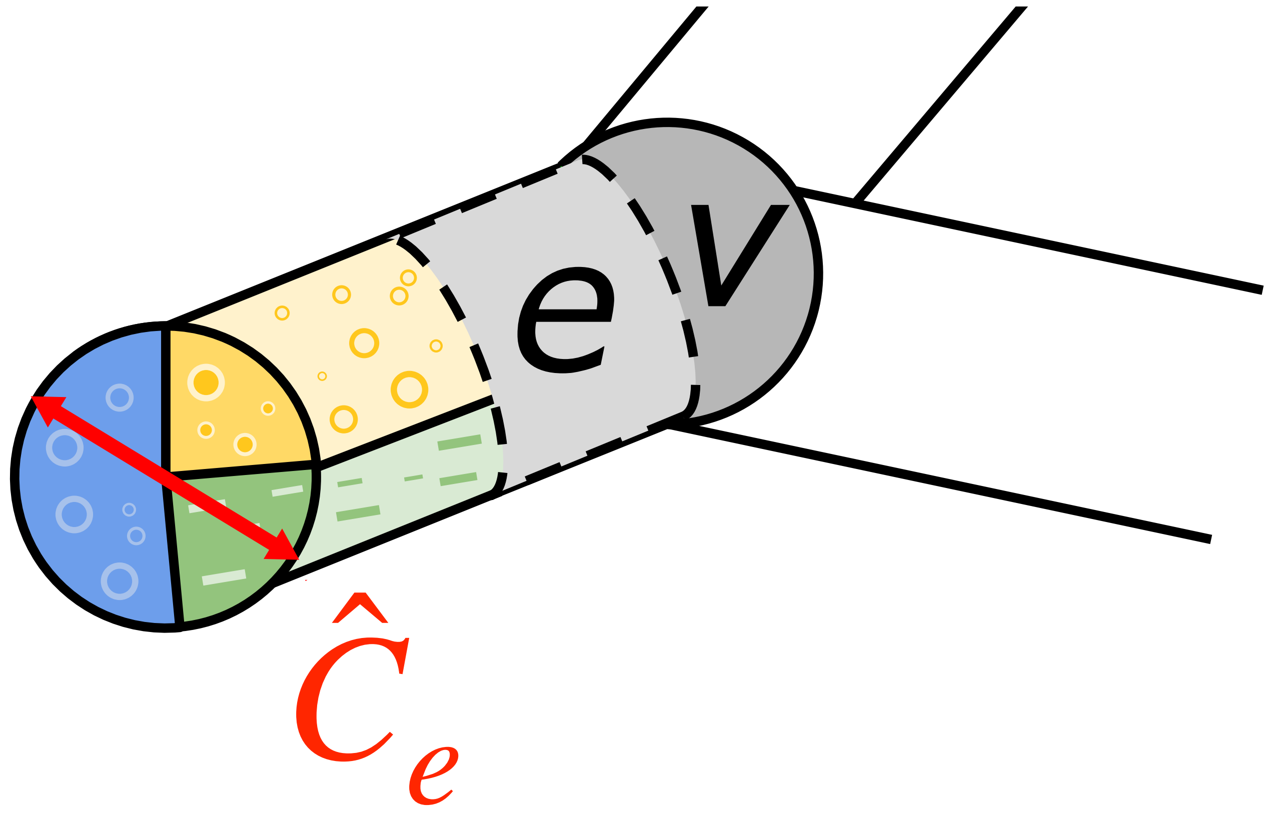
Immiscible fluids



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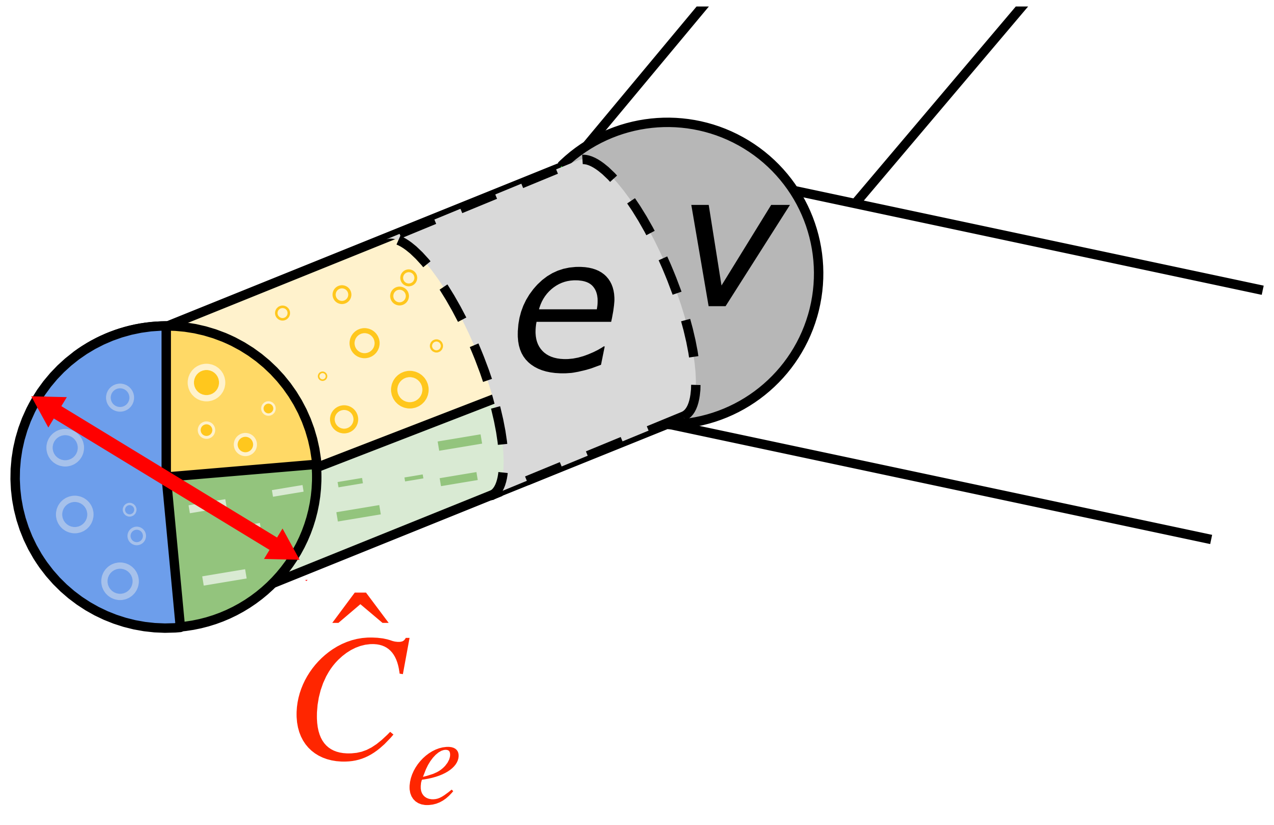
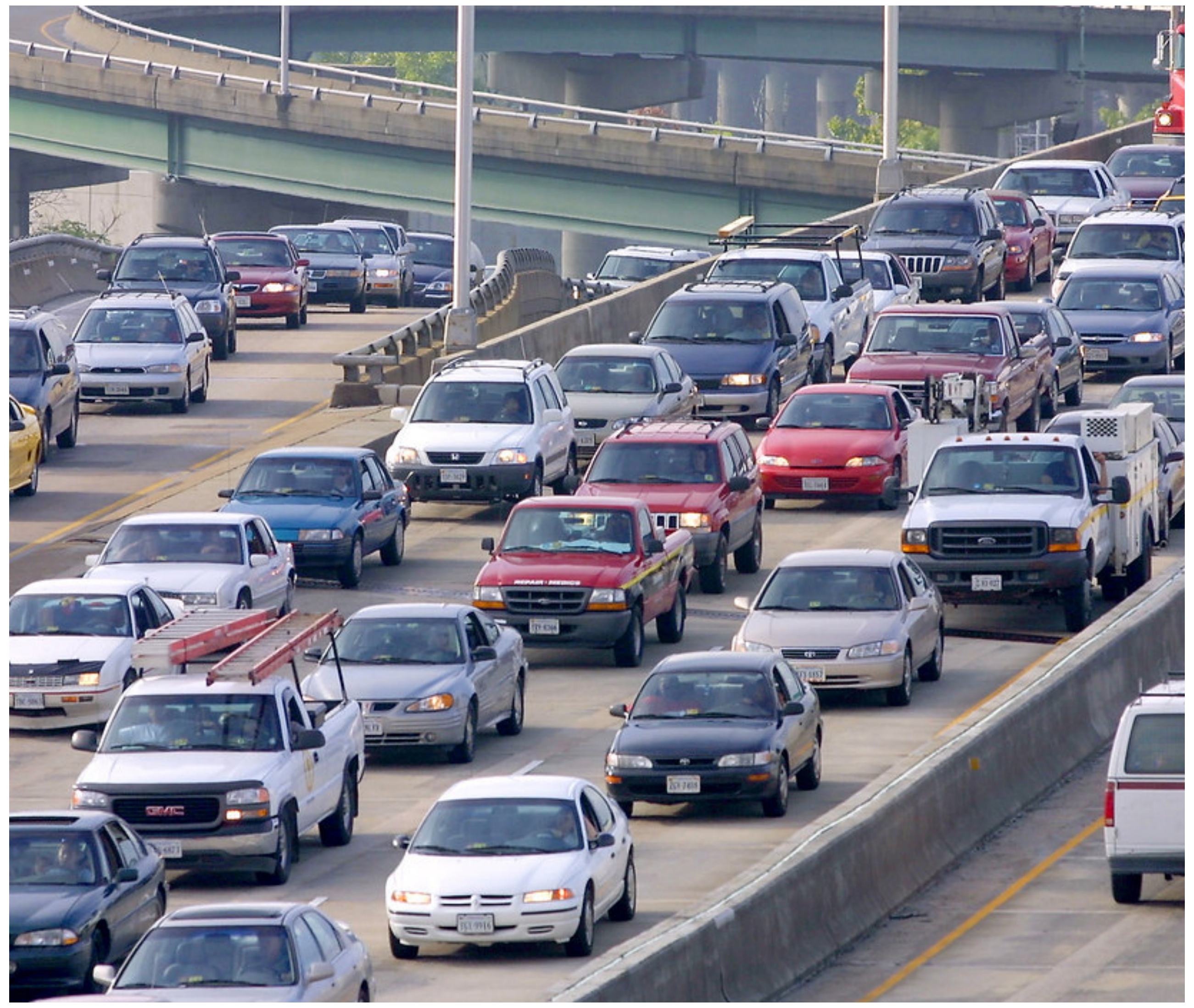
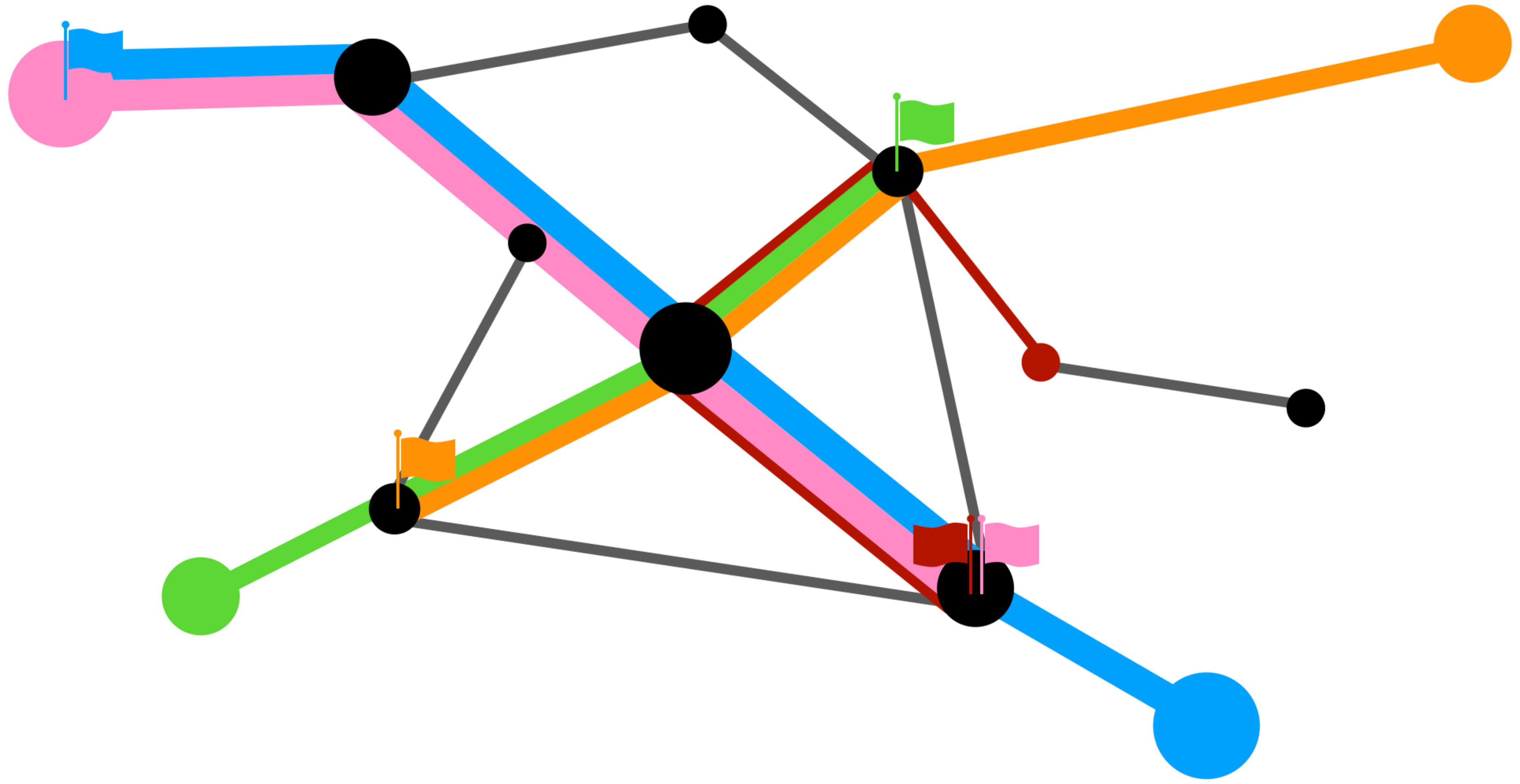


Urban transportation

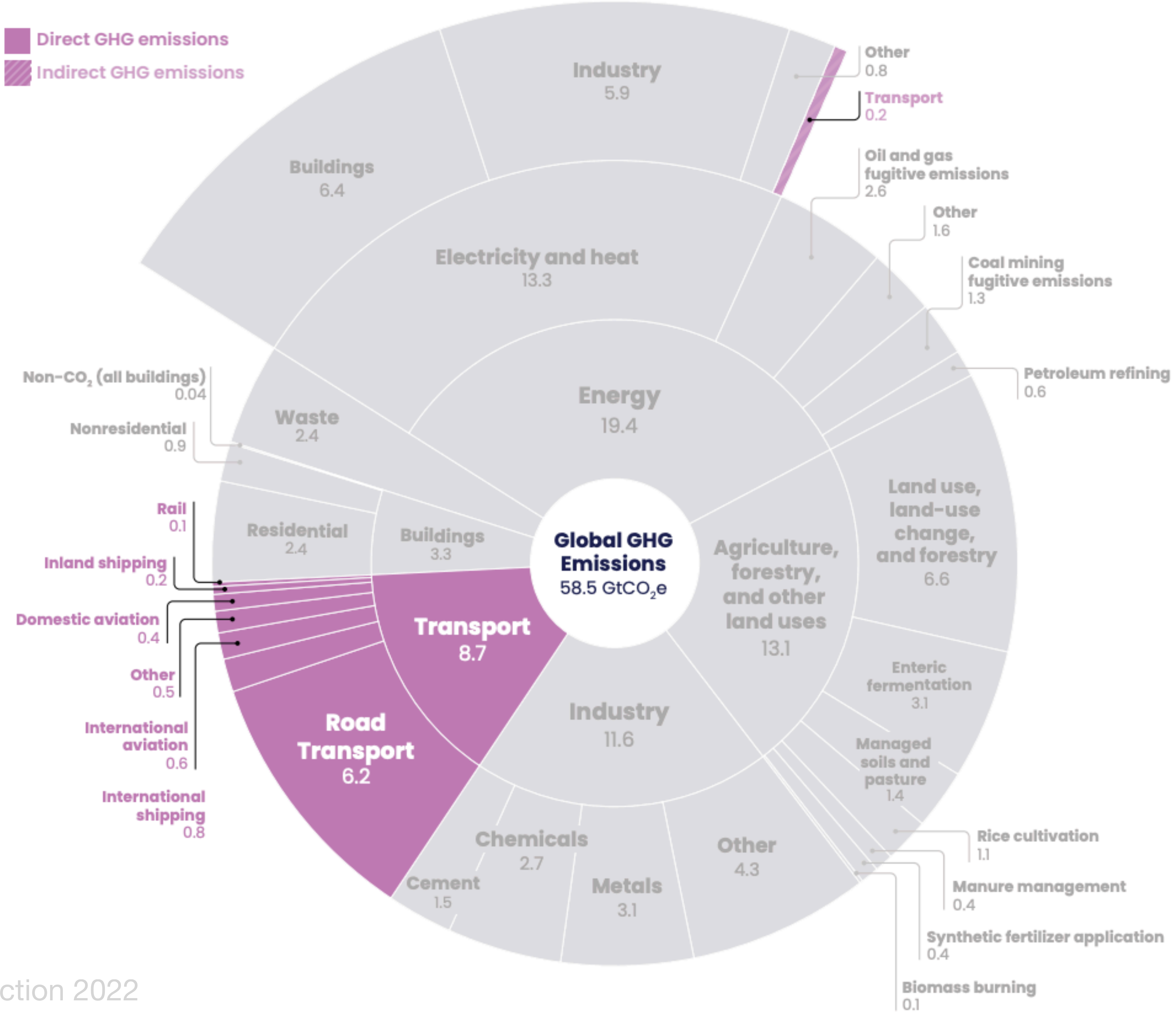




[7] Lonardi et al. PRR 2021

Why does it matter?

[8]



Data for 2019

[8] Boehm et al., State of Climate Action 2022

Multicommodity optimal transport

$$\frac{d\hat{C}_e}{dt} = \hat{C}_e^{\beta-2} f(\overrightarrow{F}_e) - \hat{C}_e$$

Computationally efficient method

[7]

$$J := \sum_e \ell_e f(\overrightarrow{F}_e(\hat{C}_e))^{\Gamma(\beta)}$$

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Multicommodity optimal transport

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Coupling between commodities: $f : \mathbb{R}^E \to \mathbb{R}_{\geq 0}$

[7] Lonardi et al. PRR 2021

Multicommodity optimal transport

$$\frac{d\hat{C}_e}{dt} = \hat{C}_e^{\beta-2} f(\overrightarrow{F}_e) - \hat{C}_e$$

$$f(\overrightarrow{F}_e) := ||\overrightarrow{F}_e||_2^2 = \sum_i (F_e^i)^2$$

[7]

$$J := \sum_e \ell_e f(\overrightarrow{F}_e(\hat{C}_e))^{\Gamma(\beta)}$$

1) Established multicommodity adaptation & minimization framework
2) Robustness of network topology is influenced by commodity coupling

[7] Lonardi et al. PRR 2021

Multicommodity optimal transport

[9]

$$f(\overrightarrow{F}_e) := ||\overrightarrow{F}_e||_1^2 = \sum_i |F_e^i|^2$$

(cost takes sum of passengers in common edge)

[9] Lonardi et al. Sci. Rep. 2022

Multicommodity optimal transport

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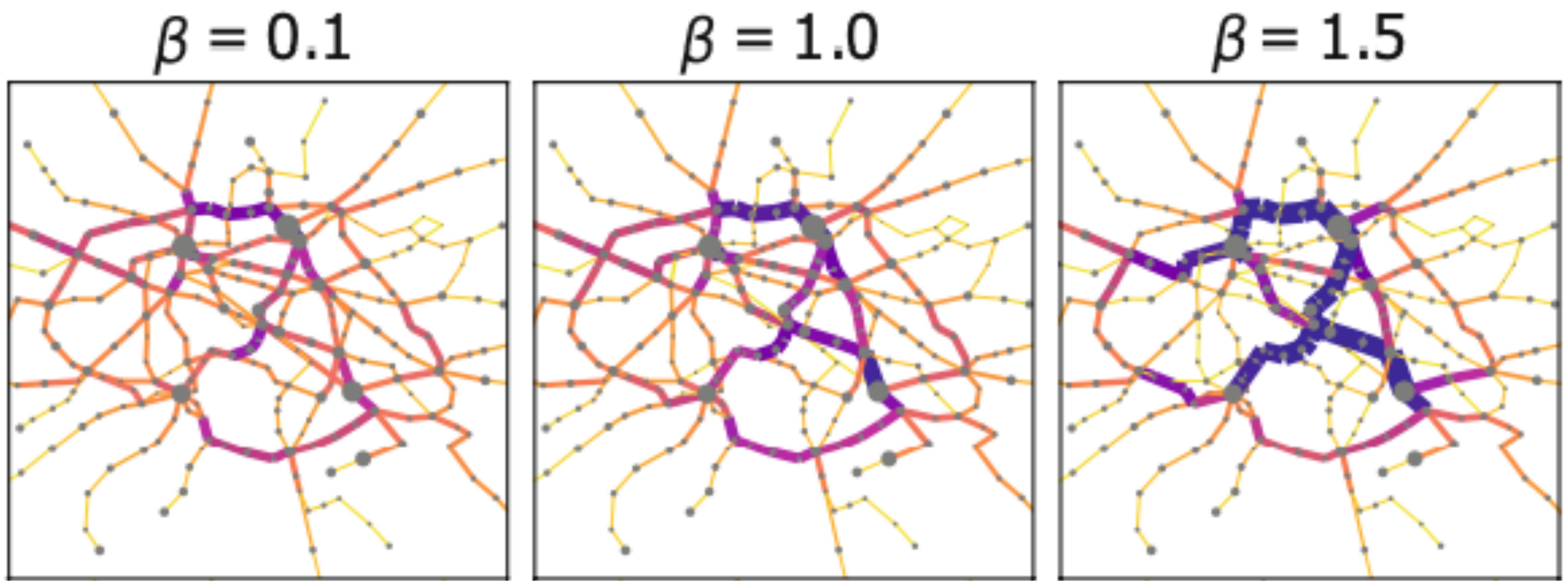
Paris Métro



Effect of β on traffic congestion

[9] Lonardi et al. Sci. Rep. 2022

Multicommodity optimal transport

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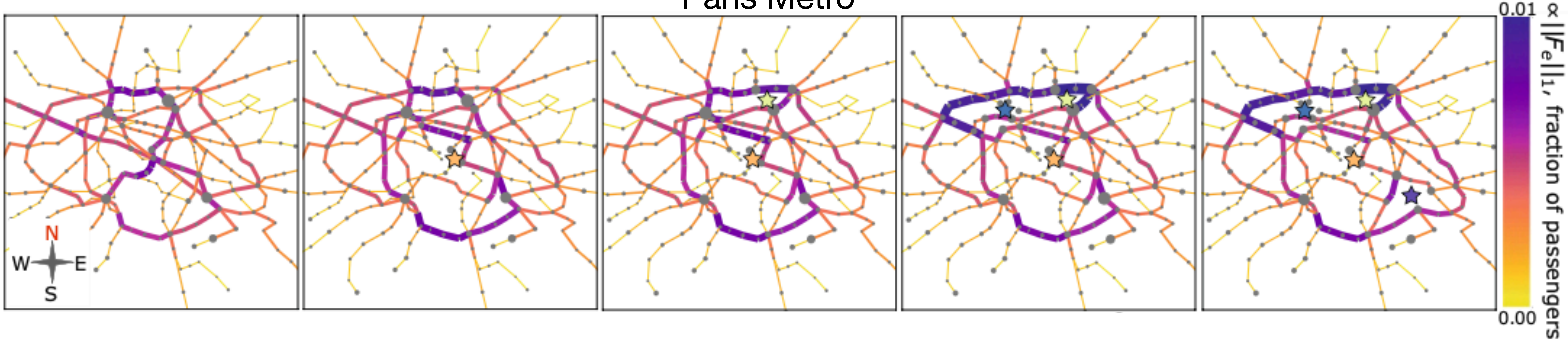

[9] Lonardi et al. Sci. Rep. 2022

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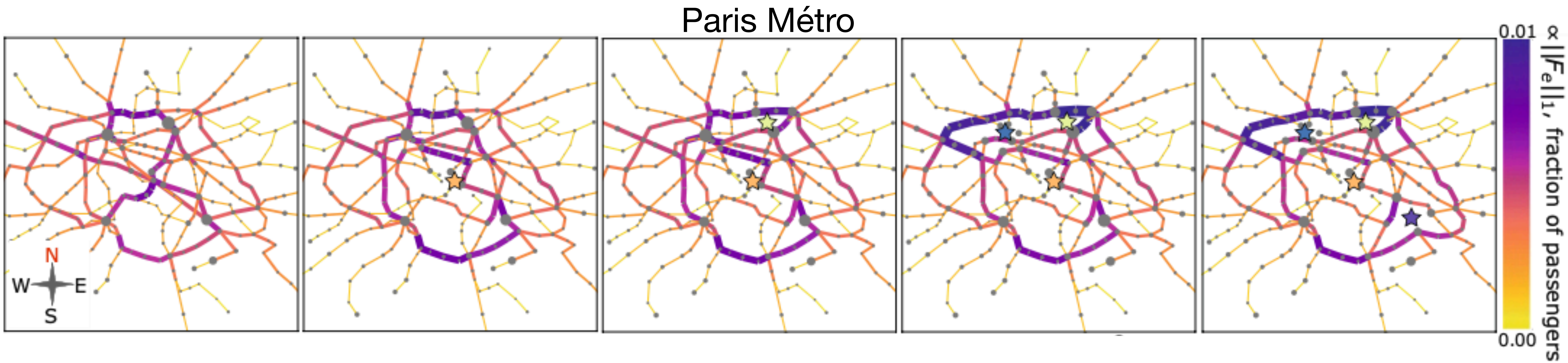


1) Prediction of transportation patterns on the Paris Métro
2) Analysis of traffic congestion, fault tolerance
3) Comparison to Dijkstra algorithm for network routing

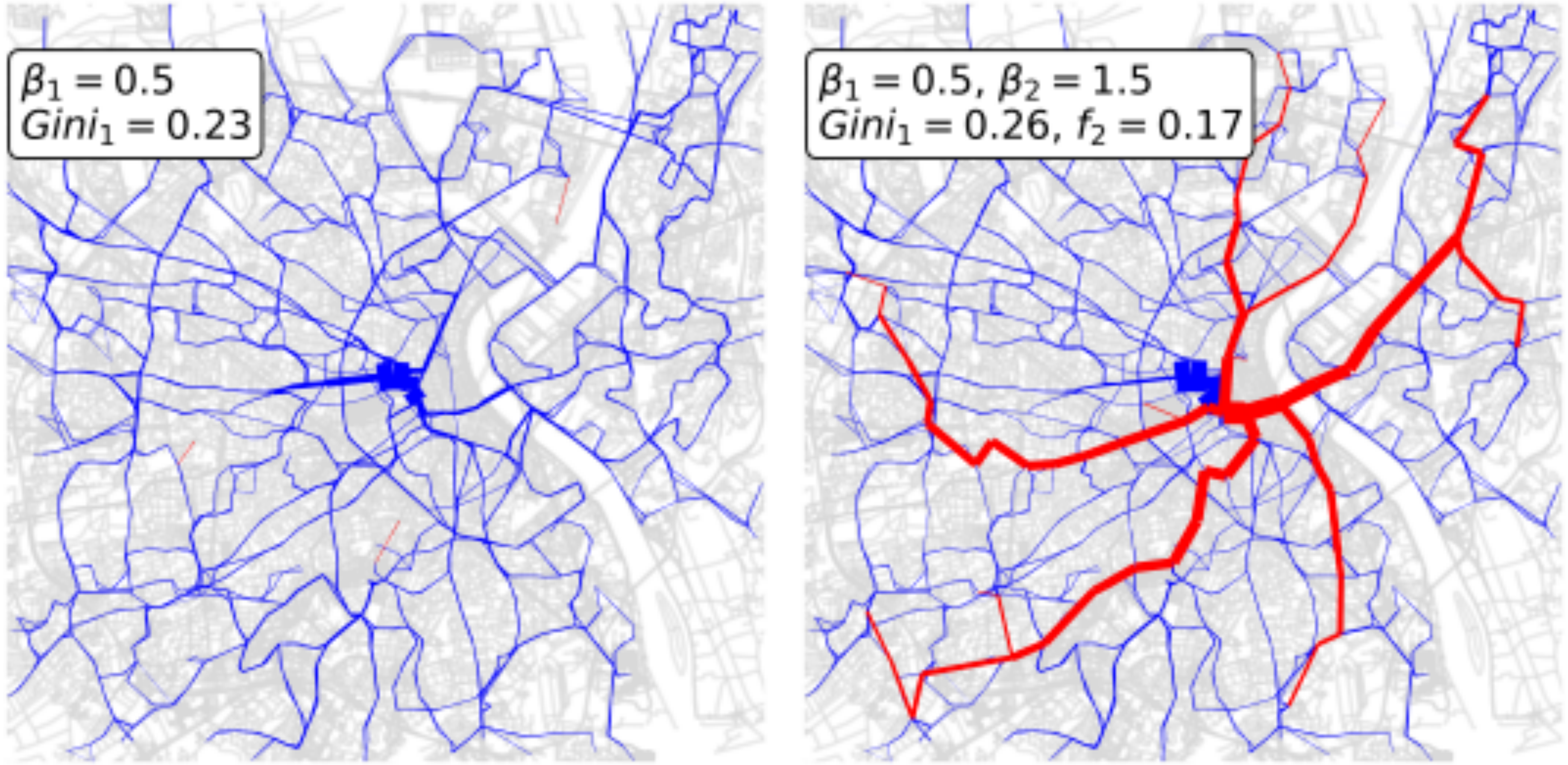
[9] Lonardi et al. Sci. Rep. 2022

Multicommodity & multilayer optimal transport

[10]

$$||\vec{F}_e||_2, \ell_e(\alpha), \beta(\alpha) \qquad \alpha = 1,\ldots,L\ (=2)$$

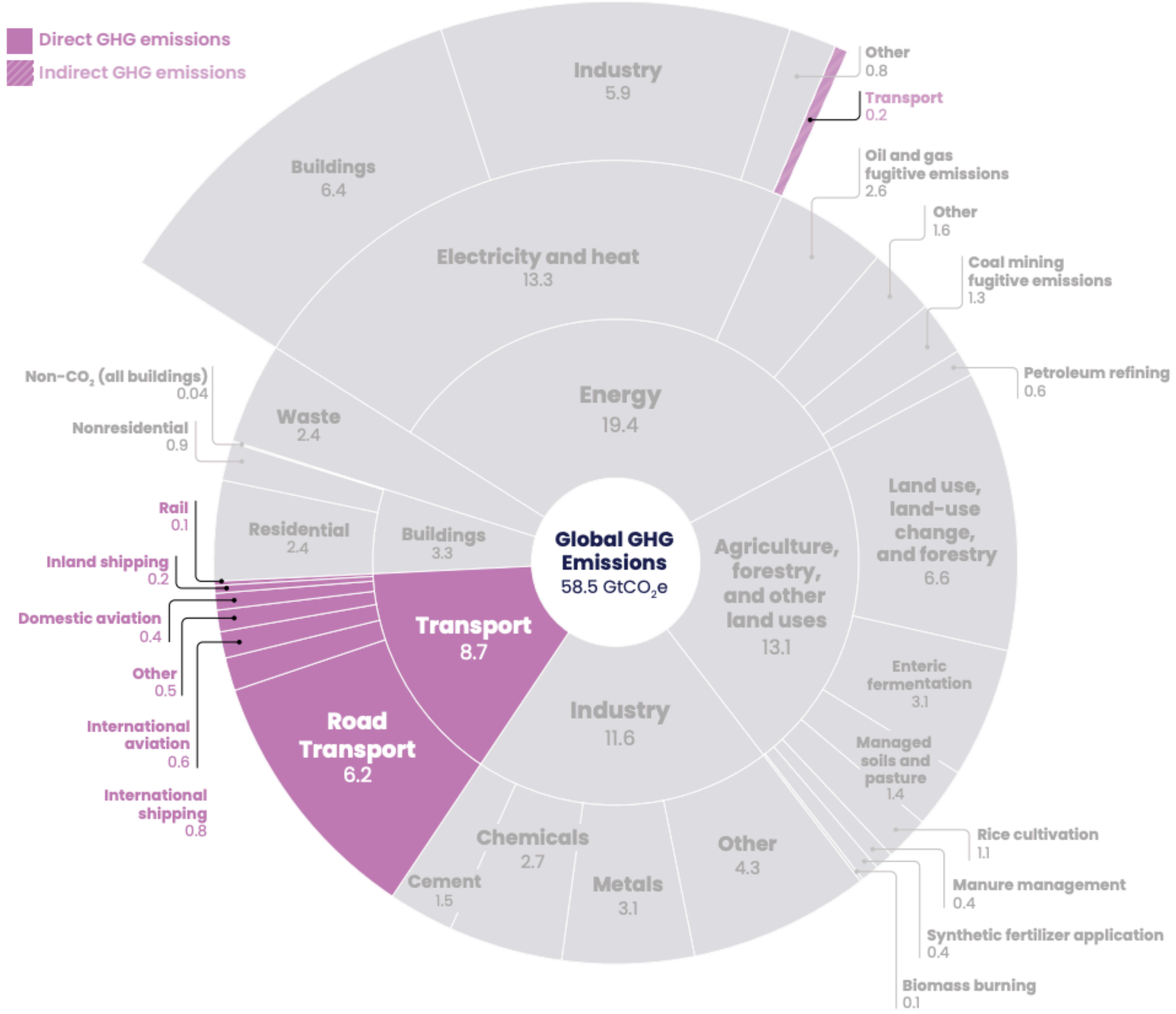
Bordeaux tram and bus



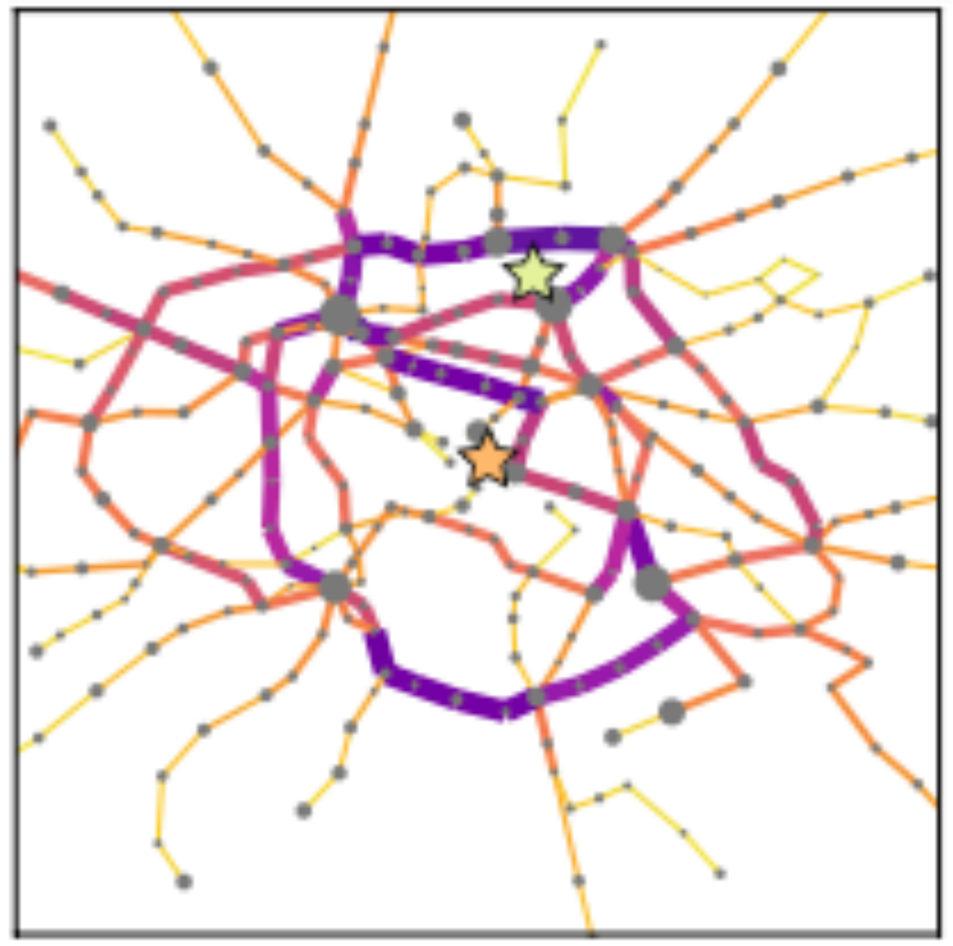
1) Tram eases traffic on roads as a measure to aid sustainability

[10] Ibrahim, Lonardi, De Bacco Algorithms 2021

Conclusions



$$F_e \rightarrow \overrightarrow{F_e} = (F_e^1, \ldots, F_e^i, \ldots, F_e^M)$$



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2) How does our algorithm compare to others?

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4) How do we translate our results in policies that can be used in practice?

[…]

Thank you!

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